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CONTENTS:

PAGE

THE ELECTRONIC THEORY OF MATTER.
WILLIAM BENJAMIN SMITH 321

PURPOSE AS SYSTEMATIC UNITY.
RALPH BARTON PERRY 352

THE ORIGIN OF TAOISM.
KING SHU LIU 376

THE CONTRIBUTIONS OF PARACELSUS TO MEDICAL SCIENCE AND PRACTICE.
J. M. STILLMAN 390

THE ORIGIN OF THE MUTATION THEORY.
HUGO DE VRIES 403

THE MANUSCRIPTS OF LEIBNIZ ON HIS DISCOVERY OF THE DIFFERENTIAL CALCULUS. Part II. (Continued.)
J. M. CHILD 411

LIBRA: The Eternal Balance of Good and Ill (Poem).
HARRY LYMAN KOOPMAN 455

CRITICISMS AND DISCUSSIONS.

Logic and Psychology. PHILIP E. B. JOURDAIN 460

The Cal-Dif-Fluk Saga (Poem). J. M. CHILD 467

Notes on De Morgan's *Budget of Paradoxes*. DAVID EUGENE SMITH 474

BOOK REVIEWS AND NOTES 478

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VOL. XXVII JULY, 1917 NO. 3

THE MONIST

THE ELECTRONIC THEORY OF MATTER.[1]

"Wie Alles sich zum Ganzen webt!
Eins in dem Andern wirkt und lebt."
—*Goethe.*

THE subject of the considerations that follow is proposed as the sixth under the division of physics in the published program of this congress. Unquestionably the proposal was most timely and fortunate, for no theme of purely scientific content is more important or more central on the stage of interest or more worthy of the attention of the assembled savants of two continents. Surely it is eminently appropriate that the New World should foster the New Knowledge, should master it, acclaim it, proclaim it, and advance it.

The most obvious criticism upon any attempt to treat this theme on the present occasion would seem to be that the barrel is too large for the hoop. So far and wide reaching is the new doctrine of matter, so interpenetrative of so many remote and alien disciplines, that any half-way adequate presentation of even the most near-lying considerations would necessarily pass swiftly beyond the largest bounds to be assigned this paper and easily expand into a stately volume.

[1] This paper, read (in Spanish) at the First Pan-American Scientific Congress (Santiago de Chile, Dec. 25, 1908 to January 5, 1909), has appeared thus far only in the *Trabajos del Cuarto Congreso Cientifico (1° Pan-Americano)*, Vol. V, pp. 1-22, Santiago de Chile, 1910.

UMI

We must begin then with renunciation. The attempt can not be made to detail but only to suggest some of the proofs (which are manifold and decisive) of the actual existence of the corpuscle, sub-atom, or electron, as the uniform elementary constituent of the visible universe is variously named. The isolated independent subsistence of this corpuscle is the central revelation of the new knowledge. It was first discovered many years ago, and proclaimed to the world as the fourth or radiant state of matter by Sir William Crookes, after whom the vacuum tubes in which the green phosphorescence accompanying the passage of an electric current were and still are named. That something called cathode rays emerged from the cathode or negative pole and moved in right lines, was proved by the shadow cast by an interposed mica cross. The English declared these rays were particles, shot out from the cathode (pole) against the inside walls of the tube; but the Germans held it was only ether waves stirred up at the pole and propagated rectilinearly. That the English were right was shown conclusively by subjecting the rays first to magnetic and then to electric attraction, whereby it appeared that they behave in all ways precisely as minute particles laden with negative electricity. Amazing is the control which the experimenter exhibits over these flying hosts of electric atoms; by deft manipulation of his infinitely fine magnetic or electric fingers he may turn the stream of corpuscles as he will and even bend it into a spiral or into a circular hoop far more supplely than one might bend the superfinest Damascus blade. But inconceivably more delicate still is the touch of the mathematical reason, whereby even the individual electron is caught in its flight and forced to tell the secret of its speed. For one may subject the flying particles simultaneously to opposite electric and magnetic influences by immersing them in two coexistent and mutually annulling fields of

force, so that they fly undisturbed straight from the negative to the positive pole. These two self-destroying forces are $He v$ and eX, whence $v = X/H$, whereby v the velocity of the corpuscle is known when we know H the magnetic and X the electric force, both of which are readily measured. This velocity increases with the exhaustion of the tube from eight thousand up to one hundred thousand kilometers per second, which is many thousand times the mean speed even of hydrogen molecules at the highest temperature ever yet attained.

But far more wonderful and incomparably more important than this determination of a variable velocity is the determination of a constant, the most fundamental yet discovered in nature. Science itself may be defined as the eternal search for invariants amid the eternal flux of variants, and this astounding constant of which I am about to speak reminds us indeed of Plato's unwavering axis of the universe turning forever in the lap of Necessity. In the equation $Hev = Xe$, the symbol e denotes the negative electric charge borne by the individual corpuscle. If we take away the magnetic force, leaving only the electric, this latter will bend the path of the flying corpuscle as gravity bends the path of the level-aimed cannon ball into a parabola. Now Galilei has taught us the formula for the amount (s) of the bend or the fall in the time t; it is $s = \frac{1}{2}at^2$ where a is the acceleration in question. Here the acceleration is the force Xe divided by the mass m of the corpuscle; the time is the tube length l divided by the velocity v; and the distance s is the descent of the green spot at the end of the tube;

hence $s = \frac{1}{2}X(e/m) \cdot (l^2/v^2)$, whence $e/m = (2v^2s/l^2X)$, where all on the right side is known. Hereby is determined this ratio of the electric charge to the mass of the flying corpuscle, and this ratio is found to be everywhere the

same (unless indeed the velocity of the corpuscle, of which it is in strictness a function, approaches that of light).

The value of this remarkable constant (for all ordinary velocities) is in the accepted C. G. S. system no less than 17,000,000 (1.7×10^7). Why is it so large? Is it because the charge e is so great, or because the mass m is so small? This question can be answered and has been answered by the exquisitely beautiful experiments of the two Wilsons (C. T. R. and H. A.) on the formation of clouds by condensation of vapor around nuclei. Not only does the water collect around particles of dust but also around any particles charged with electricity: nay more, it refuses to collect except around nuclei until the vapor reaches eightfold saturation. Now it has been found possible to free a cylinder of air from dust, and supersaturate it with vapor, and then to form in it suddenly a dense cloud by electrifying its particles with radiations from radium or still better by charging its individual molecules with electrons shot out from a metal plate played on by ultra-violet light. By attracting electrically these drops coagulated around these molecules one may suspend them in the air of the cylinder like balloons or make them fall as slowly as one will, so that their velocity of fall may be measured; and Stokes has deduced the formula for this velocity, $v = 2/9.ga^2/\mu$ where g is the known acceleration of gravity and μ the known coefficient of viscosity; hence a, the radius, and thence the size of the drop is found; and hence by measuring the amount of watery vapor deposited one finds the number of the drops and so can count the number of electrified molecules, that is the number of corpuscles, since each molecule has but one negative electron. Plainly, if by one chance in a trillion two corpuscles should light on one molecule, their mutual repulsion would dislodge them instantly.

By electrometric methods one may find the total charge

of electricity on the total water, the sum of the drops, and dividing this by the number of drops or corpuscles one finds the charge e on each, and then on dividing this by the constant ratio there results the mass m of each corpuscle. These numbers turn out to be appalling in their minuteness. The charge e equals $310/10^{12}$ of an electrostatic unit, or one one-hundred-trillionth (10^{-20}) of an electromagnetic unit, and is the long well-known approximate value of the charge borne by an atom of hydrogen in the electrolysis of dilute solutions. The mass m of the corpuscle proves to be six hundred quintillionths of a gramme (6×10^{-28}), a degree of parvitude far beyond the utmost stretch of the imagination. The same may indeed be said of the atom of hydrogen, but the mass of this atom is 1700 times[2] the mass of the corpuscle. Hitherto this hydrogen atom has been conceived as standing on the remotest confines of matter, but the new knowledge shows us a still lower world of corpuscles, nearly 2000 times smaller.

At this point it may be proper to enter a caution. It is almost universal to speak of this corpuscle as of invariable mass bearing an invariable charge of negative electricity, and the calculations and experiments do indeed yield results uniform within the limits of error. But we must remember that these experiments and calculations have always treated and apparently must always treat not the individual corpuscle but millions on millions of corpuscles and atoms. The results then were only averages of countless numbers of individuals, and the constancy of such an average implies nothing at all as to the constancy or inconstancy of the individual, just as the comparative steadiness of the rates of birth, death, marriage, homicide and the like, even in a population of a few millions, implies

[2] Later determinations raise this number to 1830 or even 1872. M. Perrin's experiments on "visible molecules" indicate that the mass of a hydrogen atom is 1.63 quadrillionths ($1.63/10^{24}$) of a gram. Hence the mass of an electron would be $0.8/10^{27}$ gram.

nothing whatever as to the rate in any particular family. For all we know the range of individuality among atoms and corpuscles may be quite as great as among suns or planets or men, and this we must say even in face of the famous dictum of Maxwell, that atoms of any one substance have all the marks of manufactured articles, being all exactly alike.

Returning from this digression we must now ask what is the mass and what is the charge of electricity of the corpuscle? It is precisely here that the new knowledge calls for the profoundest transformation of our conceptions, for it derives the phenomenon of mass in the corpuscle solely from the motion of the flying charge of negative electricity. We all know that work is needed to start a body in motion, as a car even on a perfectly smooth track. For any particular body having a particular velocity v, the amount of work necessary is perfectly definite, namely, $\frac{1}{2}v^2$ multiplied by a constant, M, called the *mass* of the body. We say the kinetic energy imparted is $\frac{1}{2}Mv^2$. This supposes the motion is in a vacuum, which is never the case; in practice the motion is always in some fluid, as water or air. Then we all know that more work is needed, according to velocity. One fans oneself gently with ease, but violently only with great effort. In fact, the fluid is also set in motion as well as the body, and this calls for energy or work. How much fluid is dragged or pressed along with the body will depend on the body's size, shape and speed and on the density of the fluid. Some of the simplest cases have been studied. Sir George Gabriel Stokes has found that in case of a sphere the work done on the fluid is $\frac{1}{2}M'V^2$, where M′ is the mass of a volume of the fluid half as large as the sphere (shown by Green, 1833), so that the total energy imparted is $\frac{1}{2}(M + M')V^2$. Now all motions take place in the all-pervading ether. It follows then that, if the ether itself has mass, when put into move-

ment it must absorb energy or require work, and hence that some perhaps infinitesimal part of the mass of a moving body even in a vacuum must be due to the swirl set up in the ether. In the case of the moving corpuscle the analogy is not absolutely perfect, but exact enough to make intelligible the statement that if a conducting sphere of radius r, having a charge e and mass m, be set moving with velocity v, the energy developed in the ethereal magnetic field has been proved to be $\frac{1}{3}k(e/r).v^2$, so that the total work done is $\frac{1}{2}[m + \frac{2}{3}k(e/r)]v^2$, and the ordinary mass m is thus increased by $\frac{2}{3}k(e^2/r)$, which stands for the inertia overcome in the ether. (This k is a factor due to the crowding together of the lines of force into a plane through the sphere center, and perpendicular to the motion, and increases rapidly as the velocity becomes great.) Since e is extremely small, this increase is wholly imperceptible in case of all single bodies subject to our senses or experiment. But for the corpuscle, when r becomes inconceivably small, this so-called electric mass assumes important proportions, yea, it accounts for absolutely all the mass of the corpuscle, which must have this electric mass and need have no other at all. For Kaufmann has measured the value of e/m (or m/e) for the various corpuscles emitted with various velocities by radium; and J. J. Thomson has calculated k for these velocities. It turns out that the calculated relative increase (due to rising velocity) in the electrical mass is constantly equal to the observed relative increase in the whole mass, whence one must conclude that the electrical mass is the whole mass, for if there were any ordinary non-electrical mass, however small, it would certainly not thus increase apparently with the increasing velocity. The mass of the corpuscle is thus not located, at least in any appreciable degree, in the corpuscle itself, but only in the universal ether around it.

Imagine a sphere surface perfectly rigid but absolutely

void, empty even of ether itself, a mere round hole in universal ether. If set in motion this hollow sphere would gather mass as it gathered velocity, but the mass would not be inside, it would be wholly outside, inwrought in the universal eddy set up in the infinite ether. In this sense the mass of the moving hollow sphere would be coextensive with the whole space filled by ether, and in this sense we may say the same of the mass of a flying corpuscle: it reaches throughout the world. We may imagine it as a mere needle-point from which Faraday tubes of force radiate to the utmost stars. But since the ether bound by the tubes varies as the squared density of the tubes, and hence varies inversely as the fourth power of the distance from the sphere center, it follows that the corpuscle mass is after all highly concentrated round the corpuscle core. For an easy reckoning shows that the corpuscle radius r is only about five millionths of the molecule radius, which is commonly taken as the hundred millionth of a centimeter; that is, of course, in *order of magnitude*. Hence from the surface of a molecule to the surface of a corpuscle at its center the mass intensity would increase more than a trillionfold.

It follows that one can no longer affirm with perfect rigor the principle of the conservation of mass, for the masses vary constantly with the velocities of the corpuscles. But to our gross senses even when armed with the most delicate instruments these variations might forever remain imperceptible. However, Heydweiler claims to have actually detected a difference in the joint weights of water and copper sulphate before solution and after, and Wallace holds that the mass of water is changed by freezing—highly interesting results that await confirmation. But it must not be supposed that the notion of mass itself is hereby eliminated or even reduced to greater simplicity. For all these results assume to start with the assumption that the ether itself has mass. Calling then to one's help

the Faraday image of tubes of force and still more the hydrodynamics of vortex rings, one may deduce from ether-mass the mass of all material bodies; but mass itself adheres along with time and space inexpugnably in our reasonings.

Corpuscles therefore are; atoms also are; how then shall we think them related? As the corpuscle mass is only $\frac{1}{1700}$ of the hydrogen atom mass, shall we think this smallest atom as compounded of 1700 corpuscles? There are many reasons against such a doctrine, reasons that lead one to think the number of corpuscles in the atom as always small. But shall we think the atom as in any case composed of corpuscles? There seems to be no escape from such a conception, which lies directly across the path of thought. For many experiments that cannot be mentioned here show that corpuscles are all-pervasive. Metals heated pour them forth, as do all other substances hot, and some shoot them out even when cold, as rubidium, and at fearful speed, as all radio-active substances; yea, if we had instruments fine enough we might detect them in every substance, and everywhere maintaining the constant ratio e/m inviolate. Moreover, that the corpuscle is closely related to the atom is clearly seen in the fact already mentioned that the corpuscle's and the hydrogen-atom's charges of electricity are the same.

Before trying to construct imaginatively the atom out of corpuscles we must recall that there are rays (as Goldstein's *Canalstrahlen*) of positive as well as of negative electricity, that are deflected by a magnet oppositely to the negative cathode rays. For them the ratio e/m is not constant and never exceeds 10,000, which is also its value for hydrogen ions in electrolysis of dilute solutions. It is natural to figure thus the positive corpuscle as a sphere of positive electrification, about the size of an atom. Of course such a sphere implies an equal and balancing amount of negative

UMI

electricity, and this we imagine distributed throughout the sphere as equal corpuscles or units of negative electricity. Since the atom is permanent, this distribution of negative electrons in the positive sphere must form a system in stable equilibrium, and the question arises, what arrangement of the electrons would constitute such a stable system? The problem is mathematico-mechanical, and its general solution lies beyond the range of our present powers of analysis; but if we propose the problem not for space but for the plane, we may solve it and get a system of answers quantitatively different but formally analogous to those that must be rendered for threefold space.

At this point theory and experiment have joined hands in a most friendly fashion. As early as 1881 the present Cavendish professor of physics at Cambridge, stimulated by the brilliant experiments of Crookes, in a long neglected but now classical paper in the *Philosophical Transactions,* discussed the motion of a charged sphere in an electric field, thereby laying the foundations of the doctrine of electric mass. Twenty-three years later (1904) he advanced to the discussion of the equilibrium of a system of negative electric charges abandoned to their mutual attractions in a positively electrified shell.* He showed that the configuration of planar equilibrium is (in general) a regular polygon concentric with the sphere, but for six particles the equilibrium is unstable, one particle will break ranks and rush to the center, while the other five form a regular pentagon. Similarly if there be seven, eight or nine; if there be ten, three will form an inner equilateral triangle, and so on up to seventeen, when one of the inner ring will again break ranks and fly to the center, leaving an inner ring of five, and an outer ring of eleven. (Four corpuscles cannot be in planar equilibrium at rest, but only when the four are in rapid rotation. At rest they are at the corners of a regular

* For an additional note see page 480.

tetrahedron whose edge equals the radius of the sphere. Six corpuscles are balanced at six corners of a regular octahedron.) When the number reaches thirty-two, the three-ring system becomes unstable, again a particle seeks the center, leaving an inner ring of five, a middle ring of eleven, an outer ring of fifteen. Looking at it another way one may ask how many must be put inside a ring of n to make it stable. The answers are: for 5, 0; for 6, 7, 8, each 1; for 9, 2; for 12, 8; for 13, 10; for 15, 15; for 20, 39; for 30, 101; for 40, 232. You see how rapidly (as the cube of n) the inside corpuscles multiply as the outer ring increases in number. The structure must be compact, densely peopled toward the center, not hollow like a shell.

The whole scheme of numbers has been worked out by J. J. Thomson (*Philosophical Magazine,* 1904) mathematically, and a beautiful experiment first made for another purpose by the American Mayer, afterwards under other forms by Wood and Monckmann, confirms his results. On a water surface any number of small equally magnetized needles are made to float by being thrust through cork discs, only like poles being above the water. These repel each other like the negative electric corpuscles. The sphere of positive electrification is represented by a magnet hung above the water, the opposite pole pointing downward. This holds the magnets in groups in stable equilibrium, and the arrangements of the magnets actually observed agree excellently with the arrangement required and predicted by mathematical analysis. Provisionally then, we may proceed on the working hypothesis that the atom is a system of corpuscles composed of a number of concentric sub-systems all held in stable equilibrium by an enclosing sphere of positive electrification. This conception may be somewhat vague and may hereafter require modification, but it is far clearer and more precise than was possible a few years ago and constitutes a notable advance in phys-

UMI

ical theory. We have spoken of the configuration as stable, but this stability must not be understood too strictly nor as always equally rigid. Since we may have an outer ring of five, six, seven or eight with only one at the center, it is plain that if there were seven in equilibrium, neither an increase nor a decrease of just one would disturb the system much; but if there were sixteen the arrangement would be one of two rings, eleven in the outer, five in the inner; take away one and the arrangement persists with only ten outside; but add one and the two-ring system is no longer stable, a particle goes to the center, the rings remain unchanged, a three-ring system is formed. A better though more complex example is afforded by the group of eleven, ranging from 58 to 68 corpuscles (Thomson); all these have five rings, thus, in order:

19	20	20	20	20	20	20	20	20	20	21
16	16	16	16	17	17	17	17	17	17	17
13	13	13	13	13	13	13	14	14	15	15
8	8	8	9	9	10	10	10	10	10	10
2	2	3	3	3	3	4	4	5	5	5
58	59	60	61	62	63	64	65	66	67	68

Here we see that if a corpuscle be injected into the 58-system it produces the least possible disturbance, place is made for it in the outmost ring and the others remain as they were, a 59-system results. But if a corpuscle be added to this it can find no place in any ring but the central; it must find its own way to the center ring or else dislodge a corpuscle from the outer ring, which corpuscle will then dislodge another from the next and so on till a corpuscle is finally dislodged into the innermost ring from the one next to it. Still another corpuscle may be injected and another and another, profoundly altering the original arrangement but preserving the outer ring unbroken.

So it goes on, the center becoming denser and the outer ring more stable till the total number 67 is reached, having the arrangement 20, 17, 15, 10, 5. Here the central mass, though it may still make room for another corpuscle, is very steady and stubborn, so that now when another corpuscle is injected, the outer ring yields, absorbs it and now has 21. Accordingly this 67-system is like that of 58, most stable, changing least from its original form. The group of arrangements from 59 to 67 corpuscles forms a series all having twenty in the outer row, the stability of each system increasing up to the last, after which a new group begins with similar properties but only eight members. Now these corpuscles are units of negative electricity. As the number of these inside the atom increases, the outer ring remaining the same, the stability, measured by Q, the work necessary to disperse all the units infinitely apart, increases; the more inside the more firmly the outside ring is held. Hence the 59-system will be least stable, a unit would easily fly off leaving only the preceding 58-system. If we suppose this 59-system neutral, on losing this negative unit it becomes comparatively at least electro-positive; in fact the most strongly electro-positive of this series of arrangements. The following members must lose more and more negative units in order to become electro-positive as this 59-system. The 67-system is charged with the utmost negative excess and so is most electro-negative or least electro-positive. The outer ring of 20 will in fact admit no more negative units inside, but on still further addition a new outer ring of 21 is formed and a new series begins with a highly electro-positive system of 68 and again runs down to an electro-negative system of 77, in each of which the outer ring is 21. Herewith then we attain a new notion of valency. For the 59-system has only just sufficient corpuscles inside to maintain its outer ring of 20; the fifty-ninth in the system, the twentieth in

the ring, might easily break away leaving only 19 outside and the atom positive from the loss of the negative unit. But it could not remain positive for it would draw to it another corpuscle and so restore its ring of twenty, and this process might be repeated. But as many as 8 negative corpuscles might be injected into the ring of 20, raising the total number to 67; hence we may say this 59-system corresponds to an atom of valency 8 for a negative unit charge and of valency 0 for a positive charge, since it could not assume permanently the positive unit charge. Consider next the 67-system. It is impossible to drive a single negative unit within the 20-ring; if one collide with the atom it stops in the outer ring making 21, but this ring is very unstable and would easily lose this electric unit; hence we may say this system has no negative valency or a negative valency equal to 0. However, this same 67-system might lose one, two or three. . . .or even eight atoms, reducing its negative, i. e., raising its positive, charge, though with harder and harder work; it could not lose more without changing its outer ring and passing into another series. Hence we may say that its positive valency is 8, just as its negative valency is 0. It is needless to dilate upon the intermediate members. Similar considerations show that we may arrange them thus:

No. of corpuscles . .	59	60	61	62	63	64	65	66	67	68
Valency .	+0	+1	+2	+3	+4	−3	−2	−1	−0	
	−8	−7	−6	−5	−4	+5	+6	+7	+8	
	Electro-positive					Electro-negative				

Such series are actually found among known chemical elements. Such are helium, lithium, barium, boron, carbon, nitrogen, oxygen, fluorin, neon, and neon, sodium, magnesium, aluminum, silicon, phosphorus, sulphur, chlorin, argon. Of course it is not meant for a moment that any

such planar arrangement is actually present in chemical atoms; their corpuscles must be arranged in tridimensional space; but it can hardly be doubted that relations analogous to the foregoing, only more complicated, must characterize spatial as well as planar arrangements. If we call the tendency of a system to shed a corpuscle *corpuscular pressure* (*outward*), then it appears that this pressure changes abruptly at the end of each series; thus at 58 the pressure is very low, at 59 very high; at 67 low; at 68 high: it falls through the electro-positives down to and through the electro-negatives. We might then define positive valency of an electro-positive (or negative) as the greatest number of corpuscules it can lose without abrupt fall in corpuscular pressure; the negative valency of the electro-negative is the number it can gain without sudden rise of corpuscular pressure. Upon these definitions and conceptions has been erected a most plausible theory of chemical combination, into which we cannot enter. But one other aspect may not be passed by in silence. While no one affirms that the planar forms of equilibrium are the actual forms assumed by corpuscles in atoms, it seems hardly possible that they are not similar thereto, similar enough to allow a most important conclusion. These forms are arranged in series, and the members of these series bear striking resemblances. There are rings outside of rings, and rings outside of these, and so on. Thus around a central one there is a ring of 6, giving 7; and around this a ring of 11, yielding 18; and around this a ring of 15 making 33; and around this a ring of 18, making 51 in all; and around this a ring of 21, which makes 72; and around this still another of 24, or a total of 96.

It seems impossible that atoms consisting of these or any such systems of corpuscles should not have many likenesses in property. We are reminded of a determinant of a definite form, whose degree is raised by border-

UMI

ing it successively by parallel lines above and below, on the right and left. The general properties of the determinants remain the same. If then all possible forms of equilibrium should actually be realized as chemical elements, of necessity these elements would fall into series and in fact a twofold series which might be expressed by a vertical and horizontal alignment, the elements in the vertical rows being alike in their central rings but differing in the number of rings; those in a horizontal row being alike in the number of rings and in their outer ring but different in their inner rings. Herewith then we are lifted up to what is commonly regarded as the apex of chemical theory, the periodic law of Mendelyeev, which thus appears not as an empirical observation, however great or important, but as an inexorable necessity of the mechanical laws of configuration and equilibrium. It is most remarkable also that herewith the law is explained not only in its rigor where it is rigorous, but also in its laxity where it is lax. For there is no necessity that all the possible forms of equilibrium should be actualized; there might very well be gaps, even considerable gaps, in both the vertical and horizontal series. In that case some gap, say in a vertical row, might have next to it some actual form in a near-lying parallel line, which would thus present not an exact but only an approximate repetition of property.

Thus the electronic theory of matter yields not only a vivid idea of the necessary existence of a double system of valences among atoms, and of the probabilities and nature of chemical combinations, but also yields deductively in a highly acceptable form the confessedly highest and most significant induction yet reached in chemical theory. This conception of the atom not as an infinitesimal grain, strong in solid singleness, as Lucretius fancied it, nor yet as a vortex-filament in an incompressible friction-

less ether, so sleek and slippery as to wriggle out from under the edge of the keenest knife and sharpest scissors, as Helmholtz and Kelvin conceived it, but as a highly organized community with members held together in unity in stable equilibrium, not at rest but in a system of complicated movements of inconceivable velocity, whose very swiftness itself contributes indispensably to the stability of the configuration (see p. 480),—this conception not only imparts new grandeur to physics but aligns it on the one hand with astronomy, on the other with biology and even anthropology. For we are all familiar in general terms with the planetary system and also with Bode's law, a special case, it would seem, of some principle of balancing, such as reaches from the atom to the constellation, from the star dust of a nebula to the most complex organization of human society. Indeed the principle of natural selection would seem to be hardly less operative in the world of brute matter than in the world of life.

More specifically, however, the electronic theory casts a broad beam of light on some long outstanding enigmas of astronomy. The motion of comets, presenting in the vast sweep of their tails an apparent repulsion from the sun, finds in this theory a long desiderated explanation. At this point science has to thank a large number of widely separated conceptions and personalities. It was the British Maxwell who as early as 1873 confirmed the suspicion of the German Euler (1746), that ethereal undulations must produce a longitudinal pressure along the ray of heat, and three years later the Italian Bartoli reached a similar, more general conclusion by a wholly different path. The mathematical prophecy declared this pressure to be $E(1 + r)/v$, where E is the energy and v the velocity of light, and r the coefficient of reflection of the receiving medium. But this phenomenon is so extremely subtile as long to have eluded the keenest observation, though the unerring finger

of mathematics was pointed at it steadily for twenty-eight years.

At length (1901) the Russian Lebedev succeeded in detecting and even in measuring it. Two years later the Americans Nichols and Hull not only repeated Lebedev's experiment with far higher precision, but showed decisively that the observed value of the repulsion agreed within the limits of error with that foretold by the English clairvoyant Maxwell. Of course this repulsive push is inconceivably minute, and on even a very small sphere it would be imperceptible in comparison with the extremely feeble attraction of gravity. However, the latter decreases as the mass or as the cubed radius, while the repulsion decreases as the surface or as the squared radius, of a spherical particle. No matter then how much greater the attraction than the repulsion on any given sphere, as the radius decreases the repulsion must finally gain the upper hand, the particle sufficiently minute must be repelled by the light away from the sun. A particle of earth 0.00001 of an inch in diameter would hang balanced between the push and the pull; if larger it would fall, if smaller it would rise and fly away before the thrust of the light. Now we need not indeed have recourse to electric corpuscles to find particles much below this critical magnitude, and the phenomenon of cometary tails blown backward from the sun with inconceivable velocity, as by the breath of a god, is readily intelligible as resulting from the now demonstrated pressure of light.

Into the details of this matter it is impossible to enter here; suffice it that the illustrious Swedish physicist, Svante Arrhenius, has subjected the whole subject to rigorous calculation which has been in the main verified, at certain points amended, by Schwarzschild, so that the enigma of cometary motion may now be regarded as solved. It is found in fact that the maximum direct pressure at the sur-

face of the sun is 2¾ mg. per square centimeter. While gravitation sets an upper limit to the diameter of the repelled particle (0.0015 mm.) the diffraction of light sets a lower, namely, about 0.1 the wave length of the light in question. Only particles whose diameters lie between these limits can be repelled, all others are attracted. The greatest theoretic repulsion is 19 times the attraction of gravitation on a particle of water; but the heterogeneity of solar light reduces this by nearly half, leaving as maximum only the tenfold of gravitation on a water sphere of 0.00016 mm. diameter. Since molecules fall far below this size, it appears that Maxwell's law does not apply to gases. But here again the corpuscle vindicates its great importance in cosmogonic theory. For the gases near the sun must be at least partially ionized, since its light is rich in ultraviolet rays and these provoke the radiation of ions. But these ionized gases, as proved by the Wilsons, are readily beclouded by the vapor condensing around the ions or corpuscles. The drops thus formed must be repelled by the light pressure, or if too heavy must fall toward the sun and bear away with them the negative electricity, leaving the gas positively laden. Hence the great part played by electricity in cometary phenomena.

While the second and third of Bredichin's groups of comets are easily understood as composed of hydrocarbon particles, the first group shows repulsion nineteen times as great as gravity, and the comets of Rooerdam and Swift (1893 II and 1892 I) even 37 and 40.5 times as great. Such repulsion would require a specific gravity only one-tenth that of water, but it may well be that hydrocarbon spherelets subjected to expulsion of hydrogen under intense solar heat may be turned into sponge-like carbon pellets of the levity required.

The combined conceptions of the forward thrust of light and the universal radiation of corpuscles give us a

UMI

widely imaginative vision of the heavens above us. The corona of the sun with its greenish pearly light is no longer a mystery. We think of it as composed of particles near the critical sizes and hence held suspended in the sky, like the coffin of Mohammed, between the pull of gravitation and the push of the light, two forces obeying the same law of inverse squares. Its tenuity transcends all conceiving, it being five million times thinner than the head of the comet, and its whole mass if of coal would be burned upon earth in a week. Nevertheless it is something, and it assures us that an endless drizzle of still finer dust is steadily poured out by the sun into surrounding space. In perhaps six thousand billion years the sun might dissipate itself entirely. Unless time be finite it would follow that at this rate all the suns in the universe would have dissolved unending ages ago and vanished like the baseless fabric of a dream. Meantime, however, there has been integration proceeding *pari passu* with disintegration. Since the earth in flight round the sun sweeps up about twenty thousand tons of meteoritic matter yearly, it is easy to reckon that the sun must catch some three hundred milliard tons in the storm that drives forever about him. These meteorites are in fact the building stones of the suns. Of what are they themselves built? According to Nordenskjöld, they are woven together of metallic atom on atom, the universal floating dust of dissolved or dissolving worlds. An awful, a tremendous cycle!

As the sun thus inspires and expires matter like some stupendous lung, similarly it respires electricity. The cloud particles repelled by light-pressure bear away negative electric nuclei and leave a positive charge behind of nearly 250 milliard coulombs, enough to dissolve 24 tons of water. As the solar dust aggregates into meteorites it dispels its negative electrons and these are drawn in a ceaseless flood toward the sun by its positive charge. The sun is thus at

once a source and a sink of electrons streaming into and from it, to and from the uttermost walls of the world.

We look aloft into the azure heavens and think to behold a sky unflecked by the minutest cloud. But reason perceives even there an eternal whirlwind of cosmic dust swirling around planets and stars and darkling suns. Well for humanity that the veil of this absorbent mist is flung abroad over the whole heavens! Else would the dome of the sky glow like a furnace, and the countenance of creation be withered and blasted. But not only do these nebulas and frozen stars shield the earth from planetary death; they are the very guardians of the universe itself (according to Arrhenius) against that heat-death (*Wärmetod*) with whose frightful specter Clausius has so long appalled the stoutest hearts of science and philosophy. For in his second law of thermodynamics the German declared that the entropy of the universe tends to a maximum while the energy remains constant. Now this entropy (of a body) is its total heat divided by its absolute temperature. It measures the amount of heat turned into a body in the process of exchange and so rendered unavailable for outward effect. The one universally observed case is that bodies at unequal temperatures tend to exchange heat till they attain the same temperature, when all effective interaction ceases. Applying this observation to the universe, Clausius declared that it was tending to such equilibrium, which would suspend all effective activity, and this universal uniform condition he named heat-death, and taught that it was inevitable, merely a matter of time — a fearful piece of reasoning, which it seemed equally impossible to accept or to refute. But the facts were against Clausius, for least of all men would he deny the past was infinite, hence the universe had had time to run down infinitely often, nevertheless it is manifestly not run down but running on still. Since it has not met

UMI

the *Wärmetod* in the infinite past, neither need it do so in the infinite future. Maxwell imagined an intelligent demon sorting out the atoms in a uniformly heated gas, by opening the windows of an invisible partition for the passage of each fast-moving particle and shutting them against the slow-moving one, thus sifting them into two apartments, the one hot and therefore able to work on, the other cold. We know, however, of no such demons, though it would be rash to deny them all existence. But on the other hand molecules moving swift enough (more than 11 kms. per second) must tear themselves away from the earth's attraction, leaving behind their slower fellows, thus sorting out the two classes after the fashion of Maxwell's demons. In this way perhaps the moon has gradually lost her atmosphere. So too may the nebulas lose fast-flying molecules that wander into their outer parts and retain only the slower ones and so remain or even become cool. Meantime the fast flyers would be caught in the widespread atmosphere of some condensing star and so raise still higher its rising temperature. Whether or no this be the exact fashion in which the universal activity is maintained, we may be sure that it is maintained in some fashion, and the dreadful presage of universal heat-death that has so long oppressed the scientific consciousness may now be dismissed as the nightmare of a fevered dream.

Still other riddles of the heavens have yielded to the divination of the corpuscular theory. Since we must now think of the sun as sprinkling all space with an incessant shower of dust, clearly the earth too must be thus sprinkled. The atmosphere can no longer be thought as imperceptible beyond 100 kms., but must certainly reach an average height of 400 kms. Were it not for the electric charges with which the particles are laden, the amount of sun-dust that could reach the earth could hardly exceed two hundred

tons yearly, one one-hundredth only of what actually reaches us in meteors and shooting stars. On the contrary, Nordenskjöld reckons the cosmic dust positively charged that reaches the earth at ten million tons yearly, and Chamberlain advocates a planitesimal theory that considers the planets as mainly built up of collected meteors. Be that as it may, the significance of the negatively laden particles in the higher regions of our atmosphere is beyond question. Of the luminous effects of wide-scattered dust in the air, the appalling eruptions of Krakatoa (1883) and Mount Pelée (1902), which reddened the sky for months, have furnished examples. On a far grander and more benignant scale the sun powders the upper air into unearthly radiance. Inhabitants of the polar regions have the nightly vision of the Aurora Borealis in two forms long confounded but now clearly distinguished: the one a nearly steady phosphorescent gleam swelling in arch on arch toward the apex of the sky; the other consisting of beams, of fountains of light spouting their torrents of splendor zenithward from the fluttering draperies of flame that fringe the northern horizon. The explanations of these two classes are of course not quite the same, and in their detail one must distinguish between the maximal and minimal years of sunspots; but in general one may say that the negatively laden particles shot out in perpetual tempest from the sun must beat upon the atmospheric envelop of the earth, which is speeding on as through a driving rain. It is the equatorial regions that are full exposed to this tempest. Great however as is the velocity of the particles, they do not in general pierce through this envelop, but are caught as it were in the net of the lines of the earth's magnetic force. Round these they spin in descending converging spirals toward the poles. On impinging upon strata of air as dense as in a vacuum tube they act as in such tubes and exhaust their energy in the fitful gleams

peculiar to cathode rays. In case however of a maximal year of sunspots the velocity of the projected particles is enormously increased and they drive on nearly in straight lines in spite of the suction of the lines of force both in the sun's and even in the earth's magnetic field. Hence the Aurora may in these years descend even toward tropical regions. It is at once perceived that the relation between the two phenomena of sunspots and polar light must be intimate and so it is, but into this there is no time to enter.

Still another astronomical puzzle, not however of polar but of tropical observation, seems now in fair way toward solution. The inhabitants of even higher latitudes may behold morning and evening at the equinoxes a cone of light rising from the horizon and called zodiacal from its connection with the zodiac. It is certainly a luminous cloud of particles glittering in the sunlight. Some have thought it to be the last remnant of nebular dust still circling the sun like a ring. But according to Arrhenius both the zodiacal light itself and its still more perplexing *Gegenschein* are due to the same corpuscular storm that pours steadily out from the sun and down upon it from all surrounding space. Surely enough has now been said to show that the corpuscular theory has the widest-reaching astronomic and cosmogonic bearings.

May it not also have biologic significance? The problem of the origin of species on our planet has gone through a variety of phases. Linnæus held that the Infinite Ens had in the beginning created just so many species, which remained unchanged down to our own day. This rigid conception was shaken by Lamarck and others a century ago, but again restored to acceptance by the great authority of Cuvier. Finally it went down forever before the researches of Darwin and the school of evolution. More recently De Vries has detected species in the very act of transvolution, not however through the gradual accumu-

lation of infinitesimal variations, but by finite leaps or mutations establishing a new species in a single generation. Meantime the deeper problem of the origin of life on our planet has not been advanced toward solution, unless the successive recognition of one proposed solution after another as unsatisfactory may be said to be advancing. Thinkers of the highest rank still cling to the notion of "spontaneous generation," under the compulsion of such reasoning as this: there was a time, no matter how remote, when there was no life on the earth; now there is life; therefore sometime between now and then life began to be. Others however spy out another possibility, namely, that life was imported from some other planet. The illustrious Kelvin insisted that the maxim *omne vivum e vivo* was as sure as the law of gravitation and hence was driven to maintain (1871) the hypothesis that life had been borne to our planet on some meteorite, some disrupted fragment of another world. But the difficulties that embarrass the development of such a notion seem quite unconquerable. More acceptable is the modern idea of panspermia, hinted by Richter as early as 1865. In a word, this doctrine holds that the germs of life are scattered as spores throughout all the deepest abysses of space, that they are driven by light-pressure along the sunbeams from planet to planet, from system to system, on journeys that may last for thousands or ten thousands of years. The intense cold of the interstellar spaces need not chill them to death since they are unaffected by a bath of liquid hydrogen (—252° C.). They need not dry out, they need not suffer any destructive chemical change, as oxidation, on their solitary flight from world to world, since evaporation and chemical processes are suspended in that Lethean flood. They need not be too large for the fingers of the light to push before it, since they have been discovered having diameters between 0.0002 and 0.0003 mm., and are doubtless often much smaller,

UMI

magnitudes well within the grasp of the sunbeam. But how could they be lifted up into the higher regions of the planetary atmosphere, and there surrendered to the propulsion of the waves of ether? Currents of air might lift them a hundred kilometers from the earth, but could never release them from the atmospheric envelope. Here again one must invoke the omnipotence of the electron. The negative-laden sundust that kindles the Aurora fires in the upper air must also beat upon these spores and charge them with electricity. So charged they must powerfully repel each other in every direction and some would be launched outward into the depths of ether and there seized and sped onward by the impulsion of the light. Is the electric field strong enough for this action? Assuredly. An electric field of two hundred volts per meter would suffice. Such are familiar near the earth's surface, and far intenser ones must prevail in the regions of polar light.

Undoubtedly it is a most perilous voyage upon which such a germ of life sets out from the system say of α Centauri to that of our own sun. The immense majority of these ether-farers would almost certainly be lost and perish; but here and there some one would arrive after nine thousand years safe at the utmost borders of our solar world and there by chance light upon some grain of sundust in the reflection of the zodiacal light and be borne along therewith sunward within the planetary ring and even into the atmosphere of some planet, as our own, which would seize upon it with gravitation and slowly drag it to the earth. Perilous too would be the landing on the shores of Time, yet some lucky sailor would succeed and so would establish a form of life upon this planet. Surely a most tremendous conception, striking wonder and awe through the hardest heart! Amazing too it is to reflect that this prodigious idea, so carefully wrought out and bulwarked at every point by the adamantine pillars of

mathematical calculation, should have been anticipated in its grandest proportions by the brooding fancy of the pre-Christian Gnostics! For in the Naassene scriptures preserved to us by the good bishop Hippolytus in his *Refutatio omnium haeresium* we find creation described allegorically in the parable of the Sower as the sowing down of seeds from the unportrayable Godhead. Moreover in the deep-thoughted Gnostic Basilides we find repeatedly the same idea, and the new knowledge has actually adopted his favorite technical term "Panspermia" as its own, to express this most recent of astronomico-biological ideas. Life then, at least in its germs, is everywhere pulsing and throbbing throughout the universe and when the finger of time points to the accepted moment the myriad forms of life leap into being out from the teeming womb of ether. We may also see that these forms are probably nearly the same, at least closely allied even at opposite poles of the Milky Way, and man may feel his blood kinship with the tenants of the remotest world. It is impossible then to repress the suggestion that the actual forms of life, being everywhere what they are, have in themselves some deep-lying hitherto unsuspected reason for being thus and not otherwise, some reason profound as the properties of numbers or the logical necessities of the geometry of Euclid.

May one not even venture to hint that the biological import of the corpuscular theory is not yet exhausted? That in the almost endless divisibility of the atoms into sub-atoms or electrons there inheres some undiscovered connection with the pangenesis required in theories of heredity? That the mysteries of Mendelism and Mutation may yet be illuminated by flashes of electric light as in a polar sky at midnight? But even were I able, there is no time to pursue this thought further.

After all, however, what do we know of electricity? Lord Kelvin declared that in his age he understood it no

UMI

better than in his youth. The colossal theory we have been considering assumes all the fundamental and hitherto inexplicable laws of electric action. In particular, besides the ether and its wonderful properties, it assumes the law of inverse squares as the mode of all interaction of electrons. Herein of course it is perfectly justified, but the mind cannot lay the importunate query, Why does this interaction vary precisely as the inverse square of the distance? Moreover, since all this action takes place in the uniform universal ether whose regions are distinguishable only by the motions that affect them, the mind is led irresistibly to the conjecture that the problem is ultimately hydrodynamical, that all these electric phenomena must some way be thinkable as movements in an all-pervading medium. Hence then the great significance of the vortex theory developed by Thomson (Kelvin) from the central property of vortex filaments discovered by Helmholtz. Kelvin and his disciples thought to recognize the indissoluble atom in the indissoluble vortex-ring and imagined that an exhaustive doctrine of knots would yield a list of atoms or elements. The new knowledge shows indeed that the atom is not indissoluble, nay, is in some cases actually dissolving, but the vortex-ring or filament is not thereby deprived of its scientific importance. It may yet be that the sub-atom or electron is essentially a vortex in ether, and the theoretic properties of such vortices may be the observed properties of electrons. At this point then the question arises: What is the relation between the equations of electricity and the equations of hydrodynamics? Or between the electromagnetic and the hydrodynamic fields? Or, finally, between the interactions of electric units and of hydrodynamic elements? It seems hard then to exaggerate the moment attaching to such researches as those of the two Bjerknes (father and son) upon fields of force, researches both experimental and mathematical.

They have proved that the relation in question is certainly a close one, that nearly all the elementary actions assumed in electromagnetic theory may be surprisingly simulated by actual experiments on pulses in a liquid medium. True, there presents itself a queer paradox: the hydrodynamic field appears not as the direct but as the inverted image of the electro-magnetic field, attraction and repulsion in the one answering to repulsion and attraction in the other. But in spite of this perversion, the results remain highly interesting and point the way along which research must follow till something more satisfying shall be suggested or discovered.

Meantime it is no more important to see clearly the wide range and immense perspective of the new knowledge than it is to recognize unequivocally its necessary limitations. Even when we suppose the hydrodynamic analogy perfect, even were it possible to state all the facts of the material and ethereal worlds, in a word, of the world of space and sense, in terms of motion rotational or irrotational in a universal ether of whatever properties, even though the vision of the Laplacian Intelligence should thus be actualized on a scale far grander than Laplace himself ever dreamed of, it would still remain true that the real problem of the world was just as far as ever from solution. For let it be understood once for all that this problem is not even to be stated finally in terms of mass and motion, the sole concepts available in the corpuscular or any other physical theory. Mass and motion are not ultimates in human thinking, no physical concepts can be ultimates. The supreme all-comprehending fact of the world is Mind, Soul, Spirit, and the ultimates of all thinking, of all reality, must be psychical. The physical world is an idea, a sensible form, which the mind constructs at every instant by the inherent law of its own activity. It is a splendid, a glorious construct, a real construct, well worthy the everlasting

UMI

study and admiration of its own creator. Contemplating this amazing creation the Spirit beholds its own image as in a mirror, and it may even explore the depths of its own being by interpreting backward its own image; just as the mathematician may translate the analytic (algebraic) properties of his equation into geometric properties of the corresponding locus and again may interpret the geometric properties of this locus into corresponding algebraic properties of the original equation. But the equation is not the locus, nor ever can be, nor would it cease to be, nor change its properties, if the geometric construction were quite impossible. Speaking then in allegory one may declare that the psychical world is a sublime equation of infinite degree; the physical world is its majestic geometric locus, its construct in terms of time and space, mass and motion. Between the two there subsists or may subsist some one-to-one relation. Let us study the grand image with unreserved admiration and with unflagging zeal. But let us never forget that after all it is only an image, a stupendous parable. Let us never forget the great word of Goethe: *Alles Vergängliche ist nur ein Gleichniss.* Otherwise the brightest achievements of physical and physiological research may prove to be only traps for our unwary feet. Otherwise we shall surely fall into the pit of materialism; we shall mistake the significant for the significate, we shall see in the whole universe only an interplay of corpuscles, and shall talk with Cabanis of the brain secreting thought as the liver secretes bile. Such an issue would be deplorable beyond all words, it would indeed be a bankruptcy of science absolutely hopeless. As secondaries electrons are invaluable; as primaries they are absolutely worthless. The favorite maxim of Sir William Hamilton abides in full validity:

"On earth is nothing great but man;
In man is nothing great but mind."

But if even the sublimest flights of physical speculation vouchsafe us no glimpse beyond the veil, how shall we ever lift it? What gaze of reason shall ever penetrate its folds? Who knows indeed that in the nature of the case it can be lifted? For my part I should be content to think it a veil of Isis. Is there not a kind of inspiration in the thought of revelation after revelation, forever and ever, world without end?

> "Higher than your arrows fly,
> Deeper than your plummets fall,
> Is the deepest, the Most High,
> Is the All in All."

Chase without catch would indeed be disheartening, but chase with nothing more to catch would certainly be empty and uninviting. If the chase is to be eternal and yet inspiring, the quarry must be infinite, the supreme truth must be forever approachable but never attainable! In the ancient myth it is said that Egeus concealed from Theseus the secret of his birth, burying the evidence beneath a stone by the seaside. The Father of heaven and earth has perhaps secreted somewhere the proofs of the origin and nature of all things; but upon the shore of what ocean, O man of science, has He rolled the stone that hides them?[3]

WILLIAM BENJAMIN SMITH.
NEW ORLEANS, LOUISIANA.

[3] Adapted from Maurice Guérin's *Le Centaure.*

PURPOSE AS SYSTEMATIC UNITY.

I.

THE present investigation is undertaken for the sake of the light which it may throw on the problem of value. Assuming that value is a function of what may broadly be termed "interest," it becomes imperative to get at the fundamental or generic character of this phenomenon. What is that attitude or act or process which is characteristic of living things, which is unmistakably present in the motor-affective consciousness of man, and which shades away through instinct to the doubtful borderland of tropism? Both the vocabulary and the grammatical structure of language provide for the teleological categories. "Purpose," "means and end," "in order to," "for the sake of," "with a view to"—these and many other kindred forms of speech are evidently applicable to the same context. There is something in our world to which they serve to call attention. What is it?

I propose to view the matter objectively rather than introspectively. What we wish to discover is the nature of the thing, and not the nature of the consciousness of the thing. It is fair, I think, to apply the analogy of mechanism. One would not think of approaching this latter question by an examination of the consciousness of mechanism. Similarly, purpose is supposed to be a kind of happening or chain of events differing in its determination from that of mechanism. It may appear that consciousness is inci-

UMI

dental to the purposive kind of determination. But in that case we should begin with the process as a whole and work in. We should not shut ourselves up in advance to the view that purpose takes place only in the introspective stream of consciousness. We cannot, in other words, determine the role of consciousness in purpose unless we first take that view of the matter in which both consciousness and its physical context are taken into account. Behavior or conduct, broadly surveyed in all the dimensions that experience affords, can alone give us the proper perspective. We want if possible to discover what it is to *be* interested, not what it is merely to *feel* interested. What is implied in *being* favorably or unfavorably disposed to anything? It may be that it all comes to nothing more than a peculiar quality or arrangement among the data of introspection, and in that case a structural psychology of feeling, will, desire or ideation will tell the whole story. But such a conclusion would be equivalent to an abandonment of the widespread notion that purposiveness is a kind of determination of events differing from their mechanical determination. The really important claim made in behalf of purpose is the claim that things happen *because* of purpose. Are acts performed *on account* of ends? Is it proper to *explain* what takes place in human or animal life, or in the course of nature at large, by the categories of teleology? The most exhaustive introspective analysis of the motor-affective consciousness would leave this question unanswered, and to confine ourselves to the data which such analysis affords would be to prejudge it unfavorably.

It is, of course, permissible to suppose that even though a case should be made for purpose in a physical or cosmic sense, value should be limited to subjective purpose. But it is evident that the question would then be merely one of terms. If there are objective purposes as well as subjective, it would be necessary to employ some term to

designate the objects of both attitudes. It would be important to construe subjective purpose as a species of this broader genus, which would be accomplished best by taking it as a kind of valuing. Furthermore objective or dynamic purpose, if there be such, would be far too important in its bearing on the special value-sciences to warrant our disregarding it in a general theory of value.

II. NEGATIVE MEANINGS.

In looking for a clue to the meaning of dynamic or objective interest, we must free our minds so far as possible from the purely negative associations which the teleological terms have acquired. The case for teleology is prejudiced by a suggestion of anti-scientific bias, or of unscientific laxity. This is due no doubt mainly to the religious or popular auspices under which it has been advanced. The teleological hypothesis is often invoked to satisfy aspirations, to flatter human nature or to conceal ignorance. In the present controversy over vitalism the proofs of purposiveness seem to consist mainly in the indictment of mechanism. Purpose is not recommended on account of its own success, but on account of the failure of something else. When so invoked it means little more than that unknown factor, x, which is needed to complete the explanation of such phenomena as growth or organic equilibrium. It is not surprising that vitalists should be regarded as impatient scientists who cannot wait for a rigorous experimental solution, but must needs invent an agency *ad hoc*; or at best as irresponsible critics who remind plodding science of its outstanding difficulties without assisting in the serious work of overcoming them.

The party of teleology according to this view is a sort of opposition party to the real scientists, who are sobered by being in power. It is the function of this opposition party to challenge and censure, rather than to legislate

and administer. It can afford to be careless or premature because it is not in office. For itself it has no policy, but confines itself to impeaching those policies of mechanism, determinism, naturalism and experimentalism which authoritative science is patiently executing. There is doubtless a certain merit in this two-party system in science. But certainly for our constructive purposes there can be no virtue in a conception of purpose as merely not something else. If the conception is to be of any use to us it must have a positive explanatory value of its own.

Nor is it at all necessary to suppose that purpose is the contradictory alternative to some other hypothesis such as mechanism. Purpose must doubtless be *different* from mechanism if it is not to lose its identity altogether; but that it should be incompatible with mechanism, or observed only in its breach, does not follow. Through being thus regarded from the outset as an antithesis to an established and universally credited theory, teleology needlessly makes enemies for itself. There is certainly no reason to suppose in advance that teleology is less compatible with mechanism than statics with dynamics, or the atomic theory with the electro-magnetic theory of light. There is certainly an empirical presumption to the contrary. A man who goes to his journey's end *in order to* keep an engagement, does not appear to violate the law of gravitation in so doing. Let us therefore endeavor to get the positive sense of the teleological type of explanation, and let us say of its compatibility or incompatibility with the mechanical type of explanation, that that is as it may be.

III. PROVISIONAL DEFINITION.

We start with the popular supposition that there is a peculiar and specific mode of explanation, which may certainly be employed in the case of the rational conduct of man, which may probably be applied to lower forms of life,

UMI

which may for speculative reasons be extended to the cosmos as a whole, and for which the name is "purpose."

1. Let us first consider a case of *human conduct.* An off-hand provisional view of this alleged mode of explanation is afforded by Socrates's famous allusion to Anaxagoras in Plato's "Phædo." Socrates, it will be remembered, distinguishes two ways of explaining his being in prison. On the one hand it is to be explained by reference to his bones and muscles. But this, he thinks, would be an inappropriate explanation; not untrue to be sure, since bones and muscles do supply the necessary "conditions,"—but not the sort of explanation that touches the real cause of a *mind*'s acting. The second and preferred explanation is in terms of Socrates's purpose of "enduring any punishment which the law inflicts." A mind, in other words, acts for the best, according to its lights. To explain its action, therefore, it is necessary to discover what it deems best, and then to construe the particular act as an instance of that best. In the present case it is supposed that Socrates is actuated by the general principle of submission to the law, and that he has judged his remaining in prison to be what under the existing circumstances that principle implies.

Let us analyze the situation more carefully. lest we omit any essential factor. In the first place, there must be a general type of action, such as submission to law, of which a particular act, such as remaining in prison, may be regarded as an instance. In the second place, there must be an agent possessed of a stable disposition or tendency to perform acts of a certain class, under varying circumstances. The particular performances will differ according to circumstances, but they must be consistent in some respect. Then, thirdly, there must be some determinate relation between the rule or type of action and the agent's disposition. But what is this determinate relation? The

simplest alternative is to suppose that the rule of action is identical with the constant or consistent feature of the disposition. Thus we might suppose that Socrates tended under varying circumstances to submit to the law. But this will not do. For if it should happen that his remaining in prison were as a matter of fact not what the law required; if it should happen that there had been some error in transmitting the commands of the authorities, or if it should turn out upon reflection that Socrates's escape rather than his passively yielding to tyrannical oppression was more in keeping with his constitutional rights, that would not disprove his purpose to submit to law. What is necessary is that Socrates should *mean* to submit to law, or that he should *think* his act to be a case of submitting to the law. The link between the rule and the disposition is an act of interpretation or judgment. In other words, one is said to be governed by a purpose M, when M is some generalized form of action, and when one is disposed consistently to perform what one believes (whether correctly or mistaken) to be a case of M.

2. This, then, appears to be what is meant by purpose when purposiveness is imputed to the rational or reflective procedure of man. Let us now turn to what common sense would regard as a more doubtful case of purpose, the case of *animal behavior*. The differentia of animal behavior which was first remarked, was the power of self-motion. Whereas an inanimate object merely submitted to motion imparted to it from without by impact, a living thing seemed to be an original source of motion. Associated with this phenomenon was the relatively unpredictable character of the action of living organisms. What they did was so far due to internal and unobservable factors that you could not rely on their yielding in any uniform way to the operation of the external forces that you might observe or apply. Living things had a way of moving of

UMI

themselves, without any apparent cause which might serve to put you on your guard. Hence they were said to exhibit "spontaneity."

It is still customary to characterize living things in this way. Biologists describe the organism as "an active, self-assertive, living creature—to some extent master of its fate."[1] But this spontaneity or self-motion no longer serves to distinguish living from inanimate things, owing to the development of the science of energy. We should now speak of this apparent spontaneity as a release of stored energy. The organism accumulates chemical energy by the process of metabolism, and then discharges it when subjected to some kind of stimulation from without. When the discharge occurs it is out of all proportion to the stimulation. Indeed in some cases there appears to be no external stimulus at all. In any case the internal factor is so much more important than the external factor that the latter affords no safe basis for prediction. As organisms become more elaborate the discharge comes to depend more and more upon the quantity and balance of its stored energies and less and less upon what is done to it from without. But even so this phenomenon of release or discharge does not differ in principle from what happens in the case of combustion or in the case of the action of high explosives. If the behavior of living things is spontaneous in this sense, there is also "spontaneous combustion" in the same sense. In the one case as in the other we now suppose that the action would be predictable even with the utmost quantitative precision if we knew the internal organization of the acting body, as well as the character and intensity of the stimulus. It is merely a question of the relative preponderance of central over peripheral factors.

Hence we are at present inclined to look elsewhere for the differentia of life, and to find it, not in the spontaneity

[1] Thomson, *Heredity*, p. 172.

of action, but in its direction *toward* something. The explosion, we say, is blind and aimless,—indifferent to consequences; whereas life is circumspect and prophetic. Forewarned is forearmed. This is what we mean when we speak of living things as exhibiting intelligence. We do not credit all living things with intelligence; but we have no hesitation in imputing it to the higher forms of animal life, and the phenomena of instinct and tropism have led to our imputing at least a quasi-intelligence to the lower animals and even to plants.

In so far as we impute intelligence to living things, we feel the need of explaining their action in a peculiar way. The explosion is satisfactorily accounted for as a resultant of two physically existing factors, the internal organization of stored energies and the external spark or trigger. But in the case of intelligence it seems necessary or at least appropriate to refer to the sequel, to that which is merely in prospect at the moment when the action occurs. Thus a dog moves rapidly away, or gets behind some intervening obstacle, when his master takes down the whip. In so far as this implies intelligence we think of it not in terms merely of existing chemical energy and the light impinging on the optic nerve. We take account also of what is going to happen, namely the painful beating. We say that that also explains why the animal is acting as he does. Or we say that the animal is acting "in order to avoid" the beating. But since the beating which is avoided does not as a matter of fact occur, we are thus appealing to a factor which is in some sense merely possible or hypothetical. Over and above the animal's power of spontaneous motion, over and above the external action of the stimulus, there is some additional factor which refers to this mere possibility and which decisively determines the direction which the discharge takes. I do not mean to assert that this third factor cannot be traced to the previous ex-

UMI

periences of the animal. Probably it can; and this has led comparative psychologists to associate intelligence with docility, or the capacity to "learn by experience." But that is not the point. However he may have come by it, the animal is supposed at the moment of action to possess a capacity for prospectively determined action. He acts not because of what is or has been merely, but because of what *may be* by virtue of his action, or what *would be* without his action. He acts, we say, from fear of a painful whipping, or from hope of immunity. There is no way of describing either the fear or the hope, without admitting it to be the fear or the hope *of something*, which something is not upon the plane of past or present physical existence as ordinarily conceived.

If, now, we put together the results of the analysis of our two examples we shall have a provisional view of interested or purposive action. In both cases there is an organism with certain accumulated energies and certain organized propensities. In both cases there is a specific external situation which acts upon the organism and liberates the energies and propensities. So far there is nothing to distinguish these cases from such physico-chemical analogies as I have cited. But in both cases there is a third and differential factor which constitutes their purposive aspect. The act is construed by the agent in terms of something ulterior and non-existential. Socrates judges his act to be of the general type of submission to law; to the dog the whip is a sign of beating or pain-to-come, and his flight is a response "as to" pain. In both cases the agent views the situation whether by inference or association, in the light of some aspect or relation that transcends given fact; and his acting as he does is determined by his viewing it as he does.

Jennings has termed this characteristic of behavior "reaction to representative stimuli." "The sea urchin....

responds to a sudden shadow falling upon it by pointing its spines in the direction from which the shadow comes. This action is defensive, serving to protect it from enemies that in approaching may have cast the shadow. The reaction is produced by the shadow, but it *refers*, in its biological value, to something behind the shadow.

"In all these cases the reaction to the change cannot be considered due to any direct injurious or beneficial effect of the actual change itself. The actual change merely *represents* a possible change behind it, which *is* injurious or beneficial. The organism reacts as if to something else than the change actually occurring. The change has the function of a *sign*. We may appropriately call stimuli of this sort *representative* stimuli."[2]

The same general principle applies to the higher organism, Socrates. That which releases his action is a representation. His friends come to his prison and urge him to escape. Their actions and words are a sign to him of law-breaking and *as such* he resists them; or his presence in prison represents to him submission to law, and as representing that, he holds to it. Let us now refine this notion of interest or purpose by comparing it with other notions which approximate it, but in some respect fall short of it or depart from it.

IV. THE PATHETIC FALLACY.

The most familiar error regarding purpose is the so-called "pathetic fallacy." It will be worth our while to inquire just wherein the fallacy lies. Suppose that in spite of my most painstaking efforts to execute a powerful stroke, the golf ball rolls ingloriously from the tee. I then turn and rend my new driver or call down maledictions upon it. I am angry not with myself but with it. I feel resentment

[2] H. S. Jennings, *Behavior of the Lower Organisms*, p. 297.

UMI

toward it precisely as though it had meant to spite me. I virtually attribute malice to it. Now this, as my less heated partner may remind me, is unreasonable, because the golf stick really didn't mean it or do it "on purpose." It is true that in effect the stick thwarts me. The stick is a cause of my displeasure. But the error consists in imputing that displeasure to it as a motive or ground for its action. In other words, it is not sufficient for purposive action that its effect should occasion displeasure; it is necessary that this displeasure as a prospective contingency should determine the act. Or take another example. Basking in its warmth, I praise the sun and feel gratefully disposed to it. If I knew what the sun liked I would gladly reciprocate. This is an innocent error, a kind of poetic license, but error it is none the less. For I have responded to the sun as though the pleasure which its rays were about to give me had actuated the sun in shedding them; whereas this effect upon my sensibilities is accidental and in no way needed in order to account for the radiation of the sun's light and heat.

But there is also a positive implication in this criticism. My own action in each case *is* purposive. My addressing the ball, or lying in the sun, is to be accounted for by reference to the stroke or the bodily comfort that is to come. My error lies not in employing such a mode of explanation but in misapplying it. There is a human weakness, doubtless one of the major motives in religion, which prompts one to extend to all the agencies involved in any event that purposive type of determination which really holds only of one's own participation in it. In the case of one's own agency the prospective sequel does account for the act, but in the case of the other contributory agencies this explanation is out of place; or, some but not all antecedent agencies are determined by the sequel. Not to discriminate is to commit the inverse of a common fallacy.

It would not be inappropriate to term this characteristic teleological error the fallacy of *"ante hoc ergo propter hoc."*

There is a further point which this error brings to light. In so far as I like it the sun's warming my body is good. The effect of the sun's action is therefore good; and it might even be that the sun "tended" to warm my body and so to do good. But that is evidently not sufficient to make the sun's action purposive. Action resulting in, or tending to, good is not *ipso facto* purposive action. It would be purposive only provided that result were somehow *accountable* for the action. In other words we are forced to recognize the essentially dynamic character of purpose. It is not the quality of the results, whether good, bad or indifferent, that implies the purposiveness of its antecedents, but the function of that result as somehow participating in the *determination* of the process.

V. PURPOSE AND SYSTEMATIC UNITY.

1. Among the widely accepted notions of purpose or interest which we shall find it profitable to examine, the next is that which identifies purpose with *systematic unity.* This notion is distinguished by the fact that it disregards the time factor, or regards it as accidental. Purpose of this sort may characterize the world *sub specie eternitatis.* It may qualify a static whole, and appear in its mere structure or arrangement, regardless of its origin or history. It follows that the purposiveness of any given reality may be judged by internal evidence, even when it is supposed that the reality in question was produced by conscious design. A purposive object is believed, like Paley's watch, to exhibit its "designedness" in its very form. This formal, static purposiveness is identified with order, system,—the interrelation of parts in a whole. Let us first consider examples, beginning with an example in which the time factor is clearly eliminated.

UMI

An ellipse is more than a mere collection of individual points; it is a curve having a distinctive character as a whole, which may be expressed by the equation $x + y = c$. Every individual point in the curve is a value of the variables in this equation, and its position is determined according to the law by the position of the other points. Although the position of each point differs from that of every other point, there is at the same time a certain identical character among them all, namely the "$x + y$" character, or the sum of the distances from two fixed points called the "foci." To call this a unified whole means that there is a definite whole-character in terms of which all of the constituents can be described. This whole-character is the law of the parts, prescribes their positions, or, as it is sometimes expressed, "generates" them. In the case of a broken line or a curve having no equation, there is no whole-character except the merely collective aspect of the several points. In that case the parts are prior to the whole, and to speak of them as parts of the whole is therefore circular or redundant. But in the case of the ellipse the whole is prior to the parts, or comes first in the order of explanation. The parts, therefore, are said to be governed by something ulterior to them. The ellipse does not exist except in so far as all the points are in their proper positions, and yet their being so disposed is determined by the nature of the ellipse. The ellipse is then said to be the purpose which regulates the several points. Each point is determined by what is necessary in order that there shall be an ellipse.

Let us now turn to examples in which time figures as one of the internal factors of a unified whole. The whole is not in time, but time is in the whole. First let us take an example of what is commonly regarded as mechanism. Suppose a body to be moving in a straight line at a uniform velocity, governed by the law of inertia. Although each

successive position of the body is new, a certain ratio of its distance-interval and its time-interval measured from any previous position is always the same. Its kinematic history as a whole exhibits a definite character which prescribes what its position must be at each particular moment. It may in its actual behavior be construed as a realization of the principle of uniform velocity. This principle in itself is a universal or ideal entity. It does not exist except in and through the successive positions of a moving body which obeys it. And yet these positions are themselves somehow determined by it.

Let us take one more example, one that is less precise but is drawn from the context of life. Modern civilization may be said to possess a characteristic flavor, which distinguishes it as a form of life. It is conditioned by the co-presence and cooperation of a thousand factors, such as the present phase of geological evolution, temperate climate, fertility of soil, racial blend, cultural tradition etc. But these many factors *compose* something. There is a unique and simple quality which somehow supervenes when all these factors are aggregated,—a quality which is identical with none of them and yet somehow takes them all up into itself. In terms of this one quality we can construe all the various conditions as contributing this or that to it. Through it they become, not so many miscellaneous particulars, but various aspects or phases of one thing. This resultant quality, or *Gestaltsqualität,* is their purpose by reference to which they are now seen to be *for* something. They may now be understood not merely severally but collectively. There is a reason why they should be together; or, over and above that determination which accounts for each by itself, there is a determination which accounts for each in its relation to the others. But this determination springs somehow from a character which does not come into existence until after they are all in place.

UMI

These examples serve to give plausibility to the notion that is now before us. Let us analyze them more carefully. It will be found, I believe, that the notion of unity which they illustrate is divisible into two types, which I shall call "ideal" and "existential" unity. The first is based on the conception of a universal. A universal unifies its instances. Furthermore it has this peculiar relation to any instance of itself: it explains the instance, or serves as a description of it, and in that sense appears to be prior to it; but on the other hand it exists, or is exemplified only through the instance, and in that sense appears to be posterior to it. So that a case of a universal seems to be something that *is* only through itself. Interrelation is an example of ideal unity. When a number of terms possess a mutual relation exclusively, that is, when they are related among themselves as none of them is related to any term without, they compose a whole. Or they may all sustain a common relation to a term outside the group. Or they may be instances of the same set of universals where the universals are themselves interrelated.

The second, or existential, type of unity consists of the convergence or fusion of many existences into one. The unity lies not in any universal or set of universals under which many particulars may be subsumed, but in an ulterior particular. Whereas unity of the first type is intelligible or apprehended by reason, unity of this second type is sensible or is a matter of empirical fact. The several particulars work together to produce a singular result, or blend into an individuality that is directly felt. Let us inquire, then, whether either of these conceptions of unity, that of universality or that of individuality, will serve as a definition of purpose.

2. It is to be noted at the outset that purpose would be an all-pervasive feature of the world we live in. Instead of its being the exception it would be the rule. Instead of

its being a residual aspect of the world, complementary to that aspect with which the physical sciences have to do, it would coincide with that orderly and law-abiding aspect of nature of which physical science has been the principal exponent. Instead of its being the antithesis to mechanism, mechanism would itself supply the most perfect instances of it. This will doubtless serve to recommend it in the judgment of those who have a predilection for teleological monism. But such philosophers cannot escape the price of their easy speculative victory. For in so far as a conception is universal it is relatively colorless. To characterize the world as purposive in this general formal sense is to say nothing more than every scientist or materialist asserts. It does not differ from saying that it is determined and intelligible in terms of laws. Democritus and Spinoza would then be as good teleologists as Plato or Leibniz. And quite apart from its philosophical barrenness such a view would be wholly inept for the purpose of a theory of value. It would wholly disregard the peculiar or differential feature of those phenomena which biology, economics, ethics and esthetics study, and would be of no service whatever in distinguishing and coordinating these sciences. Although this pragmatic objection might be thought to justify our dismissing it, it will be instructive to discover if possible just wherein this view fails to agree with our provisional conception.

3. Unity may be thought to *constitute* purpose, or to imply a purposive origin. In other words the purpose in question may be thought of as internal to the system, or as external. When intelligible or ideal unity is thought of as itself constituting purposiveness it is evident that the common view from which the teleological terms get their initial meaning, is virtually abandoned. Consider first the simple relation of a universal to its instance. A certain given curve is, let us say, an ellipse. The universal ellipse gives

UMI

the curve its character, or serves as a description of it; while on the other hand the curve gives existence or embodiment to the general nature ellipse. There is no paradox here provided we distinguish the sort of status which a universal enjoys from the status of existence. There is a peculiar relation between a universal and its instance whereby the first qualifies the second and the second realizes the first. Now it means nothing to say that the curve exists *in order to* realize the ellipse. It simply *does* realize the ellipse. The ideal nature of the ellipse explains *what* the curve is; but it does not explain the fact *that* the curve exists. Compare the case of Socrates cited above. The purposiveness of Socrates's act lay not in the fact that it was an instance of submission to law, but in the fact that its being such in some sense accounted for its occurrence. We express this by saying that Socrates performed the act because he deemed it such. In other words, the particular case of being submissive to law which in fact ensued was a condition of its own occurrence, through being referred to as a hypothetical possibility by the mind of Socrates. To construe the curve similarly it would be necessary to impute to the curve as determining its existence some reference to the possibility of its being an ellipse; which would imply a complexity of determination for which there is here no justification.

In the case of existential unity or individuality, it is admitted that a variety does possess a unitary aspect, but it cannot be said that any term of the manifold exists *for the sake of* that unity. The peculiar flavor which supervenes upon an assemblage of historical conditions is not necessarily accountable for any of them. It is not necessary to suppose that the conditions were in any sense determined by their composing a unity. This would be the case only provided among the determining factors of each condition there were one which referred to the composite sequel;

which might, of course, be the case, but could not be argued merely from the fact of the supervening unity.

The situation is not altered if we suppose any degree or any combination of these types of systematic unity. If nature throughout observes the law of gravitation, or that of the conservation of energy, so that every bodily event is an instance of the same set of interrelated universals—if it be possible to describe everything in nature by one formula, this would not in the least imply that nature exists for the sake of realizing the formula. If the world as a whole should possess a simple flavor or quality to which every existence and every event contributed an indispensable condition, this would not in the least imply that such a cosmic *quale* determined its conditions. In short mere unity as such, whether it be a conceptual unity or a perceptual unity, does not constitute purpose. This does not prove that purpose does not involve unity, but only that its differentia must lie in something else.

4. But it may still be supposed that unity argues an *external agency* of a purposive sort, that unity is a *product* of purpose. In the first place, it is to be observed that unity furnishes an almost irresistible opportunity for the pathetic fallacy. There is a strong human interest in unity, an intellectual and practical interest in ideal unity, and an esthetic interest in existential unity. When nature is found to obey relatively simple laws, and so to be predictable and workable, the mind rejoices and praises God. When sky and sea and land compose a pleasing landscape, or when a thousand different conditions conspire to bring about the existence of fuel or food, one feels instinctively grateful. And so strong is the instinct that it creates its own object. But we may dismiss this impulse as an amiable weakness. We have already seen that the fact that a state of things is an object of interest, is no proof that that state of things is owing to interest.

UMI

A second argument for the purposive origin of unity is the argument from analogy, the argument that Paley employed in the case of the watch. A thing of the type which man makes on purpose is presumably made on purpose, if not by man then by God. There is a curious paradox connected with this argument. Man is peculiarly addicted to making machines, or things which work uniformly and automatically. That being the case those parts of nature which argue a purposive creation ought to be those parts which are most mechanical, such as the periodic motions of the stars, or the conservation of energy. A living organism differs from the typical human artefact just in so far as it is spontaneous and unpredictable; and ought therefore to be the last thing to be attributed to a creative will. As a matter of fact, however, the mechanical parts of nature are the originals of which human artefacts are adaptations and imitations. Machines are made after the analogy of nature, and their machine-like character is due to what they borrow from its independent and self-sufficient forces. Invention does, it is true, correlate these forces in new ways; but there is nothing in the principle of correlation that is new. One could not look for a prettier correlation of forces than that between the centrifugal and centripetal forces of a planet moving in an elliptical orbit. The fact is that man can contrive for his own ends physical systems which resemble those which he finds in nature. The remarkable or unaccountable thing is not that systematic unity should appear in the absence of purpose, but that purpose should have anything to do with it at all. The original mechanisms of nature are relatively intelligible, and human artefacts relatively doubtful and obscure. Purposive origination is not to be invoked as a helpful hypothesis to account for a mystery; it is itself the mystery which the mechanical laws of nature will presumably help to solve.

If the argument from analogy is to be employed at all, there is more justification for arguing from the case of nature to that of human conduct than for arguing in the reverse direction. If the hypothesis of purpose is needed at all, it is needed to explain not the existence of systematic unity in the world, but the peculiar case of human conduct or animal behavior.

Nor is the case for the argument from analogy strengthened if the emphasis is put on the aspect of utility. A systematic unity which serves human needs does not require an explanation which refers to these needs. The periodic motions of the earth evidently provide the heat and light and intervals of rest without which human life would be impossible. Their utility exceeds that of any man-made agency. But to suppose that they have come about for the sake of this, is simply to lapse into that pathetic fallacy which we have already dismissed.

5. There is one further argument from unity which deserves consideration, the argument, namely which employs the notion of *probability*. It is argued that in proportion as a coincidence is *remarkable* it must have been designed. Thus, for example, Professor Henderson has shown that the physico-chemical constitution of the natural world is uniquely favorable to life. It constitutes a maximum of fitness.

"The fitness of the environment results from characteristics which constitute a series of maxima — unique or nearly unique properties of water, carbonic acid, the compounds of carbon, hydrogen, and oxygen and the ocean— so numerous, so varied, so nearly complete among all things which are concerned in the problem that together they form certainly the greatest possible fitness. No other environment consisting of primary constituents made up of other known elements, or lacking water and carbonic

UMI

acid, could possess a like number of fit characteristics or such highly fit characteristics, or in any manner such great fitness to promote complexity, durability, and active metabolism in the organic mechanism which we call life."[3]

The author then goes on to argue that "there is not one chance in millions of millions" that all these properties should simultaneously occur, and that they should be thus uniquely favorable to life, unless we assume some general law that determines them so to be.

Now, in the first place, this appears to be a misuse of the principle of probability. It is not proper to infer a law from a single simultaneity, but only from a succession of simultaneities. If the first throw of a pair of dice happens to be a double-six, that does not prove that the dice are loaded, in spite of the fact that the chances were thirty-six to one against that particular combination. There would be ground for suspecting a partiality for double-sixes only provided in the long run this combination turned up more frequently than once in thirty-six times. The general or original physico-chemical composition of the cosmos is like a single throw of dice; the chances are heavily against it, but this proves nothing as to any determining principle over and above chance. It would be possible to make such an inference only provided it were possible to gather in the cosmic elements and throw them again. It makes no difference whatever how heavy the odds are against any particular combination, provided there is only one instance of the combination; for it is entirely in keeping with a combination's unusual or remarkable character that it should occur first. In other words, the principle of chance has to do with the frequency of a combination and not with its place in the series. Where the range of alternatives is large the first combination will always be highly improb-

[3] L. J. Henderson, *The Fitness of the Environment*, p. 272.

able; but this fact follows from the principle of chance, and cannot create a presumption against chance.[4]

The same reasoning holds of the "fitness" of the environment for life. Let us suppose life to be a constant. It will then be comparable to a die having the same number on all of its faces. The environment, on the other hand, has millions of faces only one of which matches the first die. That the two should match in any single instance is highly improbable; the chances are millions to one against it. But if it should happen that there was only one trial, its happening to be successful would prove nothing as to there being anything more than chance at work. Professor Henderson insists that the relation of fitness between life and its environment is reciprocal; but he appears to ignore this essential fact, that it is the environment which is given once and for all, while the die of life is thrown again and again. It may be argued that life agrees with its environment too often to permit one to suppose that on the part of life it is a matter of chance. But nothing of the sort can be inferred on the part of the cosmic environment because that lies unchanged upon the board. The relation of matching where one term is cast once and the other repeatedly is not a reciprocal relation. If the matching is uniformly successful, it may prove that the matcher is not trusting to chance, but it proves nothing as to the matched.

Suppose that we vary the illustration. It is a remarkable fact that a given individual likes the world just as he finds it. The world agrees with his taste. In view of the vast range of possibilities, the countless worlds that would offend him, this is prodigiously improbable. But it does not follow that the world is determined to please him.

[4] Bosanquet makes this clear when he says: "We have very small ground for being surprised at the actual occurrence of that alternative which had fewest chances in its favor; and absolutely none for being surprised at the occurrence of a marked or interesting alternative which has against it enormous odds." (*Logic*, second edition, Vol. I, p. 342.)

UMI

That would follow only provided the world came up again and again according to his taste. But, unfortunately for the argument, the world does not come up again and again, but only once. Suppose, on the other hand, that sentient beings come up again and again always liking the given world. This, then, *would* argue that the taste of sentient creatures was somehow determined with reference to their environment, and did not originate independently of it.

Even this would not prove purpose. Suppose all the impressions on a given area of sand to correspond exactly and uniquely to the feet of a certain child that is at play in the neighborhood. This would presumably not be an accident; but would be accepted as evidence that one of the terms of the fitness relation, namely the feet of the child, was the cause of the other, namely the impressions on the sand. It would be necessary, however, to distinguish this case from the relation between the same child's feet and the shoes in his closet. There is fitness in both cases; and in both cases the fitness is determined, not accidental. But in the latter case alone would one say that the fitness was due to purpose. One would not argue the purposiveness from the bare relation of fitness, or from the non-accidental character of the fitness, but from the peculiar way in which the fitness was in this case determined. The shoes in the closet are of a certain shape because of being judged or expected to fit their owner. And this might still be the case even though they should as a matter of fact fit very poorly.

6. We conclude, then, that purpose in the provisional sense adopted at the outset, cannot be said to consist in the structural unity of any system taken as a whole; nor can it be inferred from such a unity, as necessary to account for its uniqueness, maximal character, aptness or any other peculiarity. The same condition of unity might or might not have been due to purpose. It is necessary in each case

UMI

to observe the actual course of its coming into existence. In other words, purpose is not to be defined in general formal terms, any more than chemical reaction. It is not the same thing as determinateness or law in general. If there be such a thing, it consists in a particular sort of agency that appears in some cases of determination and not in others. We dissent, then, from the view that purpose is exhibited in all cases of system and unity; being exhibited most unmistakably in those realms of nature that science has already set in order, and more doubtfully, therefore, in the phenomena of life.[5] We agree with those who find purpose to be a peculiarity attaching to some parts of the existent world, most unmistakably to the behavior of man; purpose in the inorganic world being a doubtful extension of a conception derived from the datum of life.

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[5] I understand that this latter is the view to which "objective" idealists incline, as illustrated by the case of Bosanquet. Cf. his "Meaning of Teleology," *Proceedings of the British Academy*, Vol. II: "The foundations of teleology in the universe are far too deeply laid to be accounted for by, still less restricted to, the intervention of finite consciousness. Everything goes to show that such consciousness should not be regarded as the source of teleology, but as itself a manifestation, falling within wider manifestations, of the immanent individuality of the real. It is not teleological because, as a finite subject of desire and volition, it is 'purposive.' It is what we call 'purposive' because reality is individual and teleological, and manifests this character partly in finite intelligence, partly in appearances of a far greater range and scope" (pp. 8-9). This "individuality of the real" which manifests itself in the larger cosmic and historical processes, where we cannot suppose it to be designed or commanded by any finite mind, would appear to consist in systematic unity of the sorts which we have defined.

UMI

THE ORIGIN OF TAOISM.*

THE western world is apt to regard Chinese reflection as predominantly ethical. This is due largely to the fact that the system of Confucius is taken as typical.[1] But this view is misleading and requires to be supplemented. In reality the Chinese mind is fundamentally concerned for the health of the inner man, and accordingly it is more properly described as ethico-spiritual. This appears to the careful reader in the teachings of Confucius himself, and it is notably true of the mystical doctrine of Lao-tze and his more immediate followers.

Taoism is well named after the central principle (Tao) which pervades this system of thought. The original meaning of the term was "way" (path), which in the realm of

* Partial publication (Part I, revised and abridged) of thesis entitled: "The Thought of Lao-tze; its origin, content and development," presented to Northwestern University in partial fulfilment of the requirements for the attainment of the degree of Doctor of Philosophy. The whole will appear in book form in the publications of the Open Court Publishing Company.

[1] The common view is well seen in Grube when he says ("Die chinesische Philosophie," in *Kultur der Gegenwart*, I, v, p. 66, 2d ed., Berlin 1913) that "überhaupt das Chinesentum in Konfuzius seine vollendetste und ausgeprägteste Verkörperung gefunden hat....Will man die chinesische Kultur mit einem kurzen Schlagwort charakterisieren, so wird man sie als konfuzianisch bezeichnen." This is very misleading. Confucianism came to be dominant over Taoism in China partly because of the royal edict of Wu-Ti (139-85 B. C.), which exalted this thought at the expense of all other, and partly because of the universal difficulty of popularizing mysticism or adapting it to institutional life. But while Confucius has had more visible effect in China the effect of Lao-tze has been more profound. "It is not Confucianism so much as Taoism which has most profoundly influenced the Chinese mind." This statement by Chang-Tai-Yen, a noted scholar and my former revered teacher, I believe gives the real truth of the matter, and it should be carried in mind always in studying Chinese thought.

UMI

moral inquiry came to mean "norm of conduct"; in time it was narrowed to mean "the rational principle in man," and then later it was extended to signify "reason in man and reality." This transformation was brought to definite accomplishment by the real founder of the system, as I believe, Lao-tze (sixth century B. C.), who was concerned to find a metaphysical basis for his ethico-spiritual convictions and to that end hypostatized the principle of Tao. Thus a convenient analogue in western thought is Reason or Logos mystically conceived.[2]

Concerning the life of the founder we know very little in detail, and of his work we have only the Tao-Teh-king which tradition attributed to him.[3] Both the historicity of Lao-tze and the authenticity of his work have been questioned. But it is my belief that, in the existing state of our data, these doubts have been disposed of definitively by Carus.[4] Certainly the proper procedure here is first to at-

[2] The term "Tao" of course long antedates the time of Lao-tze. As early as the Shu-King its ambiguity is already evident, where it means "way" (*wan-tao,* or "royal way," as the norm to which all should conform) and also "rational part of man" (*tao-sin,* or rational heart, as distinguished from *jhren-sin,* or human heart). Herein lay the germ for the development from the moral to the definitely metaphysical. The transition was therefore from "way" to "right way of life," to "life according to reason," to life in accordance with the rational principle of all reality, including man." It was this last idea which was elaborated by Lao-tze in a world-view.

[3] The Tao-Teh-King is accessible to the English reader in the excellent translation by Carus (Chicago, 1898; [rev. ed. 1913]) where (pp. 95, 96) the brief account of Lao-tze's life, by Sze-Ma-Chien, may also be found in English translation. This account gives his place of birth, family, official connection (custodian of the royal archives and state historian) and relates an encounter with Confucius. "He practised reason and virtue" we are told, and that his teaching was directed to "self-concealment and namelessness." When he foresaw the decline of his state he left for the frontier, where the custom-house officer urged him to write a book before leaving his country. "Thereupon," concludes the account, "he wrote a book of two parts consisting of five thousand and odd words, in which he discussed the concepts of reason and virtue. Then he departed. No one knows where he died." The term Tao-Teh-King was not employed before the second century A. D., but the sayings which constitute this work were uniformly referred to Lao-tze as their author. It had been customary to name books after the writer.

[4] See his admirably judicious article, "The Authenticity of the Tao-Teh-King," in *The Monist,* Vol. XI, 1901, pp. 574-601. It is my belief that the western reader of Chinese literature is in danger of hasty conclusions from the difficulty of understanding the Chinese way of thinking. The Chinese mind

tempt to justify the tradition before rejecting it because of difficulties in the Tao-Teh-King. The determining factor in this connection must be a real understanding of that work. If it can be viewed as a unitary whole produced by a single mind, the tradition may be taken as confirmed beyond question. In considering its content systematically I will hope to show that this can be done. For the present my concern is to indicate how the thought of Lao-tze can be considered in the historical continuity of Chinese reflection, after the manner of the western treatment of the history of philosophy. To that end we must deal with it as a product of preceding thought and immediate environment and the genius of our author.

The rise of a new viewpoint in the development of thought cannot be an entirely isolated affair, however novel the addition may be. This may be safely assumed for the progress of Chinese thought as it is for that of the west. Hence one may properly expect that a system such as that of Lao-tze's in the Tao-Teh-King could not have appeared without a preceding development and that accordingly it should be studied in its historical setting.

The earliest Chinese reflection centered in the conduct of man and is embodied in the Hong-Fan, which dates back to 2205-2198 B. C. and forms a part of the Shu-King (the oldest book of China). Therein we find rules laid down for

does not move normally in the channels of discursive reasoning because it is essentially intuitionistic. Insight rather than dialectic engages their attention. Hence the westerner may too readily suspect forgery in what appears to be nonsense (cf. La Couperie, *Western Origin of Chinese Thought*, p. 124, where he shrewdly observes this). The cautious reader will bear in mind the conciseness of diction in the Tao-Teh-King as standing for thought far deeper than appears, and also that the circumstances of writing precluded fuller elaboration, as well as the inevitable errors of copyists where the thought of the text is obscure in itself. In particular it is important to pay due regard to the purity of style and soberness of thought which signalize the Tao-Teh-King in contrast with the later works of the school. A stream cannot rise higher than its source, and a forgery would necessarily have revealed those fantasies and vagaries which are so conspicuous in the writings of the later Taoists. The unsympathetic reader is apt to be robbed of insight both for seeing this obvious fact and also for getting the real meaning at the heart of the perversions and aberrations.

UMI

the ordering of one's inner life, the securing of proper balance between conflicting tendencies in one's nature, the relation that subsists between man and the natural world-order as well as that between man and his fellows. In it we find too the conception of the king as the embodiment of the eternal moral principles, the "royal way" (*wan-tao*) which was conceived of as the objective criterion to which men should conform their personal preferences. And in it we find the notion of Tao also as "rational part of man," as above indicated. The idea of Tao therefore goes very far back in Chinese thought.

In addition to the Shu-King there is also the Yih-King, or "Book of Changes."[5] Therein is outlined the first Chinese cosmological scheme, as well as an ethical doctrine based on this cosmology. It posits an original principle called *Tai-Chi*, the "Great Origin," and two primary forces called *Yin* and *Yang*. It was thought that the world was formed through the action and reaction between these two principles. A cosmos was regarded as possible only when there was a perfect balance between these two basic elements, otherwise chaos would ensue. The attendant ethical doctrine centered in the notion of moderation. As in the objective order so in man an equilibrium of opposite forces was the aim. Going to extremes was regarded as disastrous, because contrary to the course of nature. The cosmology and the ethics of the Yih-King were therefore constituent elements in Chinese reflection long before they appeared in the Tao-Teh-King.

In addition to these two sources there were probably other documents which were later lost, as the quotations in the Tao-Teh-King would indicate. Moreover, the accounts of the lives of ascetics make plain that from early times there had been men who lived in seclusion, insulated

[5] The rudiments of this work were in existence prior to the date of the Shu-King, but were not elaborated until about 1200 B. C.

UMI

from the currents of social and political life. With the advent of the period of storm and stress, at Lao-tze's time, this ascetic spirit became much intensified. It took deep hold on the thoughtful and serious-minded men of that age, some of whom betook themselves to rural pursuits while others moved about apparently without profession, eccentric and mysterious in behavior.

In the Tao-Teh-King the connection with the past is evidenced by certain expressions[6] which indicate clearly a consciousness of debt to preceding thought. This has long been recognized by Chinese scholars and has been largely responsible for the impulse to find the origin of Taoism in reflection antecedent to Lao-tze. Thus Hwang-ti, the legendary emperor of the Chinese, has been regarded as the founder of Taoism, though on very meagre evidence.[7] Again it has been suggested that Lao-tze was simply the transmitter of wise sayings and proverbs out of the past.[8] Another account makes Lao-tze to have sat under a master, Shan Yung, who was already advanced in years.[9] Still another view finds the origin of Taoism in the Yih-King, whose cosmology and ethics bear so striking a resemblance to those of the Tao-Teh-King.[10] In short, Chinese scholars have been amply aware of a continuity between preceding reflection and that of Lao-tze, and the connection is so obvious that there is danger of thereby overlooking his originality.[11]

[6] Such, for example, as "The Ancients say," "The Poet says," "The Sage says" and the like.

[7] Based on the fact that a passage of the Tao-Teh-King is quoted from a book attributed to Hwang-ti no longer extant. The same passage is found at the beginning of the work of Lieh-tze. The existence of such a book was denied by Hwai-Nan-tze.

[8] By Chu-Hsi (1130-1200 A. D.)

[9] See Hwai-Nan-tze (ch. 10) Lao-tze "learned the lesson of tenderness by watching the tongue." The allusion is to old age when the teeth have fallen out.

[10] See Yih-King, especially Books III, VI and XI, Engl. transl. by Legge (*Sacred Books of the East*, Vol. XVI).

[11] Cf. Carus, *op. cit.*, p. 31, and Strauss, *Lao-tze's Tao-Teh-King*, pp. lxiii ff.

That Lao-tze had free and full access to the literature of his day is sufficiently attested by the tradition which made him custodian of the royal archives and state historian. This included the classical literature which has survived and probably much that has since been lost.[12] It is inconceivable that a contact of this kind should have failed to influence the development of his thought. In addition there were certain records of the hermits or recluses who preceded him and to whose general circle he is supposed to have belonged. The contempt for temporal goods, the effort to create a world of their own beyond that of ordinary values, the spirit of thoroughgoing renunciation which characterized this group are essentially the marks of the thought of Lao-tze. Such influence of his predecessors and contemporaries in thought must therefore be assumed if we are to avoid the impossible idea that the construction of Lao-tze was wholly *de novo*.[13]

Thus it is clear that Lao-tze enjoyed the intellectual heritage of his age. But we must recall that this heritage reveals no such systematic character as may be found in the Tao-Teh-King. This work is so characterized by simplicity and unity, it so bears the impress of a single individual, that it suggests inevitably to the reader who has entered into its spirit a seamless fabric woven from the deeply experienced convictions of a distinct personality. One must therefore assume some genius operative in revitalizing and bringing *en rapport* with his age the inherited

[12] The Shi-King, Yih-King and Lih-King would have been accessible to Lao-tze in their ancient form and not as revised by Confucius.

[13] The possibility of foreign influence in the shaping of Lao-tze's thought, either direct or indirect, I do not consider here. Where the effort is made (e. g., by Harlez, Douglas, La Couperie, Strauss, Rémusat, in varying degrees) the proof rests upon mere resemblance in mystical or mythological or religious conceptions. Such procedure is too open to the charge of precipitate generalization on the basis of fancied resemblance and too hazardous in the absence of supporting external evidence to win more than doubtful assent. It may be true that such foreign influence did exist in fact. But the state of historical knowledge is at present entirely inadequate to furnish satisfactory conclusions. It seems therefore to me more desirable to seek to account for Lao-tze by reference to indigenous conditions.

UMI

thought of the past. As Eucken well says, with the western philosophy in mind, "It is not so much the past which decides as to the present as the present which decides as to the past, and that in accordance with this, our picture of the past continually changes, depending upon the spiritual nature of the present.[14] So in the Chinese constructive activity of the sixth century B. C., for which the historical evidence is ample, the living present served to stimulate and illuminate the obscure potentialities of the past. Certainly the writer of the Tao-Teh-King was possessed of a genius for illuminating even the homeliest wisdom in the literature and tradition at hand, and by new insight into the significance of Tao he was enabled to unfold the possibilities lying inherent in this supreme principle.[15]

But with all his genius Lao-tze was a part of his age, and hence he must be considered in relation to the conditions then prevailing. What has been so distinctly true in the progress of western philosophic thought again must be taken to maintain in its degree for the development of Chinese thought. "Philosophy," says Windelband, "receives both its problems and the materials for their solutions from the ideas of the general consciousness of the time and from the needs of society. The general conquests and the newly emerging questions of the special sciences, the movements of the religious consciousness, the intuitions of art, the revolutions in social and political life—all these give philosophy new impulses at irregular intervals, and condition the directions of the interest which forces, now these, now those, problems into the foreground, and crowds others aside for the time being."[16] Here we have the course indi-

[14] *Main Problems of Modern Philosophy,* 1912, p. 319.

[15] The unfolding of the past by synthesis of the various elements therein is perfectly familiar to the student of western philosophy in its development. It is so much a condition of progress in that thought that its history is replete with illustrations. I believe the same may be safely assumed for the development of Chinese thought, however more measured its progress is.

[16] *History of Philosophy,* 1893, p. 13.

cated which must be followed in our inquiry concerning the origin of Taoism. The rise of this system of thought must remain an obscure mystery unless we regard the environment of Lao-tze, in connection with his heritage and his genius, and seek to understand the Tao-Teh-King attributed to him in relation to the cultural *milieu* in which it arose. To this we turn now.

The first form of government that Chinese history discloses to us may be designated an elective monarchy, in the sense that the successor to the throne was chosen by the nobles and ministers. In this way Yao (2357-2255 B. C.) and Shun (2255-2205 B. C.) came to hold the imperial scepter. A change came with Yü (2205-2197 B. C.) who chose his own son to succeed him and so departed from the established mode of procedure, and who laid the basis for the feudal system by assigning portions of the empire to members of the imperial family. The exact course of the ensuing development it is impossible to follow. But with the Cheo dynasty (1122-249 B. C.) feudalism had become established as a well-defined political institution. As elsewhere in political history it consisted in dividing the empire into fiefs or estates to be distributed among the various nobles for the purpose of consolidating the empire.

This feudal system worked well at first, largely because strong emperors held the scepter of state and the fief-holders served as a bulwark to the throne. But as time went on the emperors forgot the labors of their forefathers and turned more and more away from the responsibilities of government to the gratification of personal desires. As a result of this there came about gradually a decline of the central power. The various nobles and princes, who had theretofore been kept within control, began to show signs of recalcitrancy and to assert their own powers. This process of encroaching upon the royal prerogatives in-

UMI

creased more and more until the emperor became a mere figurehead, a negligible factor, and the real power passed into the hands of the vassals. With this came a contest among the various states for supremacy, and so the nation was precipitated into a tumultuous maelstrom of strife. The balance between the forces which make law and order possible had become violently disturbed. Factional strife and internecine feuds became the order of the day. There ensued a reckless rush for self-aggrandizement and an unscrupulous disregard of rights, and brute power replaced reason. To supplement the military force the resources of craft and cunning were pressed into service and the Machiavellian attitude became dominant.

Along with the political decline went hand in hand a cultural deterioration. In place of the earlier devotion to peaceful pursuits, with its cultivation of arts and literature, there arose an exaggerated emphasis upon material values, and the earlier simplicity was supplanted by sophistication both in thought and in action. In this rule of unreason the complex social organization, which the first few rulers had succeeded in building up, had completely collapsed. At the beginning of the dynasty, especially in the reign of Chen-Wang (1115-1079 B. C.), there had been worked out an elaborate system of etiquette, which in point of complexity has no parallel in history.[17] But in these troublous times this fell to pieces. Neither the weaklings on the throne nor the contending vassals were inclined to maintain this elaborate system. And where all forces were working for disintegration naturally all phases of the social life were affected. The established ethical standards also broke down to be superseded by personal whim and caprice. No-

[17] In its ramifications it extended to every phase of social and political life. Regulations were prescribed even for such details as mode of dress, eating, toilet, form of address, etc., etc. Its apparently immutable and fixed character testifies to the genius for organization of its author, Cheo-King, and also accounts for the fascination which it exercised over the mind of Confucius later who felt impelled to refer to that period as the great age of culture.

where could universal rules of conduct be found, as in the ancient days. Unjust laws were enacted in place of the old regulations, which had been so nicely calculated to promote orderly life. The life of the people was made miserable by all sorts of oppressive measures, and their very life-blood was drained that the craving of the rulers for military glory and the excitement of the chase might be satisfied. In short, a condition of affairs existed which was strikingly similar to that which prevailed in France prior to the Revolution. Wherever one looks he is confronted with unreason and disorder resulting from the chase after worldly gain and the abuse of power.

Such were the conditions prevailing in the world into which both Lao-tze and Confucius were born. The intensity of the crisis may be measured by the fact that China's two greatest creative thinkers arose at this time, after whom really significant thought in that country continued to develop. The system of each was adapted to solve from its angle the problem set by the aggravated situation. Confucius was conservative and sought to reconstruct in harmony with the past, while Lao-tze was radical and could be satisfied with nothing short of complete breach. Each may be conceived as crystallizing the spirit and thought of the type which he represented. The temperament of the one was essentially institutional and accordingly gave itself to reconstructing the social fabric as existing, as is abundantly clear out of all his writings. The temperament of the other was wholly impatient with all temporal expedients and would not stop short of permanent peace in some eternal principle; this he found by reconstructing the ancient Tao as supreme principle of men and reality, as also amply appears in his work, the Tao-Teh-King.

The contrast between the two men was really antip-

UMI

odal'[18] and by reference to it the signficance of the genius of our author stands out at its highest. Confucius was characterized by moderation and sanity as the world of common sense measures these qualities. In his efforts at reform he confined himself wholly to the attainable, in conformity with the sagacity of the plain man. His keen sense for concrete reality forbade him to step forth with anything like a Utopian program. He clung to the solid ground, with never a desire to soar in the empyrean realms. He was no doctrinaire, no mere theorist in any sense, but a practical reformer. To mend the situation as he saw it he set about to abolish the feudal system, as the source of disintegration, and to reestablish the monarchy with its stabilizing force of imperial power. To counteract the forces that were making against law and order he set out to revive the doctrines of the ancient sages, the system of Cheo-li, whose exact and rigid orderliness very naturally fascinated his type of mind. Hence his supreme emphasis on ritual and his belief that the golden age lay in the past.

But the spirit of Lao-tze was radically different and permitted no such direction as that of Confucius in his solution of the problem. His genius impelled him to make a clean sweep and led him to a very different reconstruction. He felt that the world had gone so far astray that it could not be reformed by mere revival of ancient traditions or by any other patching-up process. He demanded some radical procedure, a complete reversal of the existing order. He felt deeply the insecurity, nay, the utter collapse of the foundations of life in his age, and he sought a basis so

[18] This contrast is revealed in beautiful simplicity in the report by Sze-Ma-Chien concerning the interview between the two men (Carus, *Tao-Teh-King*, pp. 95, 96). The difference between these men is vividly portrayed by Grube, who writes: "Auf der einen Seite ein Mann, der mit beiden Füssen auf dem Boden der Wirklichkeit steht und....nur nach dem Erreichbaren strebt. Auf der anderen Seite das Wolkenkuckucksheim eines einsamen, weltfremden Denkers. Dort zielbewusstes Streben nach staatlicher Reform auf sittlicher Grundlage, hier asketische Weltflucht und mystisches Versenken ins ewige Tao."

secure that it might not be shaken. Like Plato, so much in this his fellow-spirit of the Occident a century and a half later, he regarded the present order as wholly bad and not to be compromised with. And like Plato he turned away from the immediate world of strife to the life of reflection and contemplation, to find a world that was characterized by the eternal as opposed to the temporal. But more mystical than Plato he found his solution by way of the inner life and communing with nature. In revolting against the existing order he was driven to withdraw from externals like the true mystic that he was. And in so withdrawing he found within his inner self the supreme principle of his own and of all being. Thus he was enabled to give new life and meaning to the doctrine of Tao, as a simple and unitary principle of all reality.

To this abiding principle he called his wayward people to return. In opposition to the spirit of self-assertion that pervaded the age, he called for complete renunciation, for the surrender of the petty ambitions of the ego which only in this way could realize Tao. Instead of the feverish and scattered haste so common in his day, he enjoined quiet confidence in the fundamental reason of the universal order. Against over-regulation and the multiplication of laws and statutes he therefore went the full length of a doctrine of *laissez-faire*. He would have none of the ceremonies and rules of etiquette on which the conciliating Confucius later laid such stress; they were for him the most prolific source of the great evil of hypocrisy, being merely external show. All parading of virtue or even conscious well-doing was for him an evil. He would eliminate all virtue except that of acting according to Tao and all knowledge save that of Tao. This was the sum and substance of his thought. And the solution which he disclosed to his age as the way of salvation was an unfolding of this.

But Lao-tze did not stand alone in this negative atti-

UMI

tude toward the existing order of things. He was a true spokesman for those fellow spirits of his race and day who had also turned unreservedly to the inner life for refuge from the storm of the external world. Like all great leaders of thought, our philosopher gave form and body to the longings and aspirations in the minds of the many less gifted. He is clearly the concentrated embodiment of the quietistic and mystical spirit of the recluses already referred to. They were in need of a spokesman to make clearly articulate what they felt and experienced, and this was supplied by Lao-tze. As the genius of Confucius enabled him to serve as a constructive guide for the type he represented, so the genius of Lao-tze enabled him to create for and direct the less numerous but relatively widespread number of the opposite type.[19]

Such then was the place of Lao-tze in the origin of Taoism. He was its real founder because it was his genius that established it. What had grown up during long centuries and undergone gradual transformation was brought by him to articulate formulation under the impulse of an environment which pressed to a mystical solution. His fundamental doctrine was the long familiar Tao, but its central position and multiple unfolding in man and in reality required the labor of genius for establishment. Lao-tze was that genius, and so Chinese history has recorded

[19] In the Confucian Analects alone reference is made to fourteen such recluses who ridiculed the effort to reform a decadent society. The fortuitous character of these meetings and the fact that they are recorded by Confucius and his disciples attest how widespread the movement was. Strauss (*op. cit.*, pp. xliii ff) has suggested the ingenious theory that there was already in existence a Taoist sect (*Tao-Gemeinde*), whose teachings were reduced to writing by Lao-tze. There is no basis in fact for this conjecture, and it overlooks the real ability of Lao-tze. But this is undoubtedly a more correct direction for interpretation than that which disregards the widespread nature of the movement.

In this connection it is of great importance to bear in mind, contrary to a too prevalent misconception, that even Confucius had to give up his efforts at reform in despair in his later years, and that he was forced to content himself with the more quiet work of teaching and of editing books. The real significance of his work lay in this preparation for posterity rather than in his actual effect on his own age.

him as one of its two great creative thinkers. Accordingly his doctrine, as set down in the Tao-Teh-King, is found to exhibit the unity and simplicity which signalize that work. It is essentially the reaction to a most difficult situation of a born mystic who was able to give full expression to the mysticism of his people. And what has been said of the mystic in general maintains for Lao-tze in an eminent degree. "What the world, which truly knows *nothing*, calls 'mysticism,' is the science of *ultimates*,. . . .the science of self-evident reality, which cannot be 'reasoned about,' because it is the object of pure reason or perception."[20] Herein is contained the key to the true understanding of Lao-tze's work.

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[20] Quoted from Patmore by Underhill (*Mysticism*, 4th ed., 1912, p. 29).

THE CONTRIBUTIONS OF PARACELSUS TO MEDICAL SCIENCE AND PRACTICE.

THERE appears to be little doubt as to the real value of many specific contributions of Paracelsus to medical knowledge and practice, although competent authorities differ widely as to the extent and character of his influence upon medical progress. It may be admitted that his vigorous assaults upon the degenerate Galenism of his day were effective in arousing an attitude of criticism and questioning which assisted greatly the influence of other workers whose labors were laying less sensationally but more soundly the foundation stones of scientific medicine.

Vesalius, often called the founder of the modern science of anatomy, and Paré, the "father of surgery," were both contemporaries of Paracelsus, though their great works appeared only after the death of Paracelsus. The "Greater Surgery" of Paracelsus had appeared nearly thirty years before Paré's classical work and had passed through several editions, and it is said that Paré acknowledged his indebtedness to Paracelsus in the preface to the first edition of his work.[1]

Admitting that none of the medical treatises of Paracelsus has the scientific value of the works of his great contemporaries, it should nevertheless not be forgotten

[1] Cf. Stoddart, *The Life of Paracelsus.* London, 1911, p. 65.

that his work may have had an influence for progress in his own time much greater than its present value in the light of later knowledge. Dr. Sudhoff records some nineteen editions of the "Greater Surgery" by the close of the sixteenth century, in German, French, Latin and Dutch languages, and other works of his shared in somewhat less degree in this popularity.

The disapproval and hostility of the universities and the profession toward Paracelsus should not be permitted to mislead us into underrating his influence, as it may be recalled that both Vesalius and Paré also suffered from this hostility. Vesalius was denounced by his former teacher Sylvius as an insane heretic and his great work on anatomy was denounced to the Inquisition. Though he was not condemned by that body his professorship at Padua became untenable, and he was forced to return to his native city Brussels and is said to have become a hypochondriac as the result of his persecutions.

Paré was more successful in maintaining his professional position through official support though the faculty of the University of Paris protested his tenure of office.

The history of medical science and discovery has been the subject of more thorough study than most of the natural sciences, and a number of competent critics of early medical history have estimated the place of Paracelsus in the development of various departments of that science. From such sources may be best summarized the contributions of Paracelsus.

Thus with respect to surgery, Dr. Edmund Owen in the *Encyclopaedia Britannica* (eleventh edition, article "Surgery") says:

"The fourteenth and fifteenth centuries are almost entirely without interest for surgical history. The dead level of tradition is broken first by two men of originality and genius, Paracelsus (1493-1541) and Paré, and by the re-

vival of anatomy at the hands of Andreas Vesalius (1514-64) and Gabriel Fallopius (1523-1562), professors at Padua. Apart from the mystical form in which much of his teaching was cast Paracelsus has great merits as a reformer of surgical practice. It is not, however, as an innovator in operative surgery, but rather as a direct observer of natural processes that Paracelsus is distinguished. His description of hospital gangrene, for example, is perfectly true to nature; his numerous observations on syphilis are also sound and sensible; and he was the first to point out the connection between cretinism of the offspring and goitre of the parents."

So also Proksch,[2] the historian of syphilitic diseases, credits Paracelsus with the recognition of the inherited character of this disease and states that there are indeed but few and subordinate regulations in modern syphilis-therapy which Paracelsus has not enunciated. Iwan Bloch also attributes the first observation of the hereditary character of that disease to Paracelsus.[3] That Paracelsus devoted so much attention to the consideration of these diseases was evidently made a subject of contemptuous criticism by his opponents as may be inferred from his replies to them in the *Paragranum*.[4]

"Why then do you clowns (*Gugelfritzen*) abuse my writings, which you can in no way refute other than by saying that I know nothing to write about but of *luxus* and *venere*? Is that a trifling thing? or in your opinion to be despised? Because I have understood that all open wounds may be converted into the French disease (i. e., syphilis), which is the worst disease in the whole world,—no worse has ever been known,—which spares nobody and attacks the highest personages the most severely—shall I

[2] Quoted by Baas, *Geschichtliche Entwickelung des ärztlichen Standes*, p. 210.

[3] Neuburger und Pagel. *Handbuch der Geschichte der Medizin*, III, 403.

[4] Paracelsus, *Opera*, Strassburg Folio, 1616. I, 201-2.

therefore be despised? Because I bring help to princes, lords and peasants and relate the errors that I have found, and because this has resulted in good and high reputation for me, you would throw me down into the mire and not spare the sick. For it is they and not I whom you would cast into the gutter."

Dr. Bauer[5] calls attention to the rational protest of Paracelsus against the excessive blood-letting in vogue at the time, his objections being based on the hypothesis that the process disturbed the harmony of the system, and upon the argument that the blood could not be purified by merely lessening its quantity.

"For the healing art and for pharmacology in connection therewith," says Dr. E. Schaer in his monograph on the history of pharmacology,[6] reform is in the first instance attached to the name of Theophrastus Paracelsus whose much contested importance for the rebirth of medicine in the period of the Reformation has been in recent times finally established in a favorable direction by a master work of critical investigation of sources. . . . But however much overzealous adherents of the brilliant physician may have misunderstood him and have gone at times beyond the goal he established, nevertheless the historical consideration of pharmacology will not hesitate to yield to Paracelsus the merit of the effective repression of the medieval polypharmacy often as meaningless as it was superstitious and to credit him with having effectively called attention to the pharmacological value of many metallic preparations and analogous chemical remedies."

Dr. Max Neuburger[7] thus summarizes the claims of Paracelsus to a place in the history of the useful advances in medicine:

[5] *Geschichte der Aderlässe*, 1870, p. 147.

[6] Neuburger and Pagel, II, 565-6.

[7] Neuburger and Pagel, II, 36ff.

UMI

"Under the banner of utilitarianism Paracelsus rendered the practical art of healing so many services that in this respect his preeminent historical importance cannot be doubted. In bringing chemistry to a higher plane and in making the new accessory branch useful to medicine, in comprehending the value of dietetics, in teaching the use of a great number of mineral substances (iron, lead, copper, antimony, mercury), and on the other hand in teaching the knowledge of their injurious actions; in paving the way to the scientific investigation of mineral waters (determination of the iron contents by nut galls), in essentially improving pharmacy (with his disciples Oswald Croll and Valerius Cordus) by the preparation of tinctures and alcoholic extracts....he has achieved really fundamental merit for all time."

It was also no unimportant service that Paracelsus rendered to medical science in attributing to natural rather than to the mystical influence of devils or spirits such nervous maladies as St. Vitus' dance. It is doubtful perhaps if his influence in this direction was very immediate upon contemporary thought, at least if we may judge from the sad history of the trials, tortures and executions of witches during a century after the activity of Paracelsus.

Doubtless also the fantastic character of the philosophy of Paracelsus itself served to diminish the effect of his sounder and saner thought.

A distinguished student of the history of science, Andrew D. White, thus characterizes the services of Paracelsus in this direction.[8]

"Yet in the beginning of the sixteenth century cases of 'possession' on a large scale began to be brought within the scope of medical science, and the man who led in this evolution of medical science was Paracelsus. He it was who first bade modern Europe think for a moment upon the

[8] *History of Warfare of Science and Theology*, II, 139.

idea that these diseases are inflicted neither by saints nor demons, and that the 'dancing possession' is simply a form of disease of which the cure may be effected by proper remedies and regimen. Paracelsus appears to have escaped any serious interference; it took some time, perhaps, for the theological leaders to understand that he had 'let a new idea loose upon the planet,' but they soon understood it and their course was simple. For about fifty years the new idea was well kept under, but in 1563 another physician, John Wier of Cleves, revived it at much risk to his position and reputation."

An interesting thesis maintained by Paracelsus was the doctrine that every disease must have its remedy. The scholastic authorities had pronounced certain diseases as incurable, and they were accordingly so considered by the profession. Rejecting as he did the ancient authorities, Paracelsus naturally enough rejected this dogma as necessarily true. Manifestly also he believed that he himself had with his new remedies effected cures of certain of these diseases, though he makes no pretension to be able to cure all diseases. The history of medical thought and discussion shows that this thesis of Paracelsus was a frequent subject of partizan debate during the century after Paracelsus.

Paracelsus sustains his thesis, however, not by the method of modern science—upon evidence of experiment and observation—but by the philosophical or rather metaphysical argument of its *a priori* reasonableness in the divine purpose, and by his interpretation of the doctrines of Christ.

"Know therefore that medicine is so to be trusted in relation to health—that it is possible for it to heal every natural disease, for whenever God has entertained anger and not mercy, there is always provided for every disease a medicine for its cure. For God does not desire us to die

but to live, and to live long, that in this life we may bear sorrow and remorse for our sins so that we may repent of them."[9]

"There is yet another great error which has strongly influenced me to write this book,—namely, because they say that diseases which I include in this book are incurable. Behold, now, their great folly: How can a physician say that a disease is incurable when death is not present; those only are incurable in which death is present. Thus they assert of gout, of epilepsy. O you foolish heads, who has authorized you to speak, because you know nothing and can accomplish nothing? Why do you not consider the saying of Christ, where he says that the sick have need of a physician? Are those not sick whom you abandon? I think so. If then they are sick as proven, then they need the physician. If then they need the physician, why do you say they cannot be helped? They need the physician that they may be helped by him. Why then do you say that they are not to be helped? You say it because you are born from the labyrinth [of errors] of medicine, and Ignorance is your mother. Every disease has its medicine. For, it is God's will that he be manifested in marvelous ways to the sick."[10]

This is obviously setting dogma against dogma, and opposing to scholasticism the methods of scholasticism. Yet that this dictum of Paracelsus was not without influence upon contemporary thought is evidenced by a passage in the writings of Robert Boyle in the century following.[11]

"Though we cannot but disapprove the vainglorious boasts of Paracelsus himself and some of his followers, who for all that lived no longer than other men, yet I think

[9] Paracelsus, *Liber de religione perpetua.* Sudhoff, *Versuch einer Kritik,* etc., II, 415.

[10] Par., *Op.* I, 253. "Die erste Defension."

[11] Boyle's Works, Birch's ed., I, 481.

mankind owes something to the chymists for having put some men in hope of doing greater cures than have been formerly aspired to or even thought possible and thereby engage them to make trials and attempts in order thereto. For not only before men were awakened and excited by the many promises and some great cures of Arnaldus de Villanova, Paracelsus, Rulandus, Severinus, and Helmont, many physicians were wont to be too forward to pronounce men troubled with such and such diseases as incurable and rather detract from nature and art than confess that these two could do what ordinary physick could not, but even now, I fear, there are but too many who though they will not openly affirm that such and such diseases are absolutely incurable, yet if a particular patient troubled with them is presented, they will be very apt to undervalue (at least) if not deride those who shall attempt to cure them."

His rational consideration and treatment of wounds and open sores is worthy of note. Instead of the customary treatment of closing up by sewing or plastering, or covering them with poultices and applications, he advocated cleanliness, protection from dirt and "external enemies." and regulation of diet, trusting to nature to effect the cure. "Every wound heals itself if it is only kept clean."[12]

There is no doubt that Paracelsus enjoyed a considerable reputation as a skilful and successful practitioner, and there is contemporary testimony, as well as his own statements, to show that he was frequently sent for even from long distances to treat wealthy and prominent patients whose maladies had baffled the skill of the Galenic physicians.

It is of course true that popular reputations of physicians are not always the true measure of ability even in our day. Nevertheless there seems little reason to doubt in spite of the assertions of hostile critics of his time, that

[12] Cf. Helfreich in Neuburger and Pagel, III, p. 15.

with his new remedies, his keen observation, and his unusually open mind, he was indeed able to afford relief or to effect cures where the orthodox physicians trammeled by their infallible dogmas were unsuccessful. That his new methods sometimes did harm rather than good is quite possible. That would naturally be the result of breaking radically new paths. And an independent empiricism—a practice founded upon experiment and personal observation seems to have been his practicce and his teaching, "*Expe-rentia ist Scientia.*" It seems probable that in his dealings with the sick, his fantastic natural philosophy was rather subordinated to a native common sense and practical logic. As stated by Professor Neuburger (*op. cit.*, II, 35), "We see in Paracelsus....the most prominent incorporation of that enigmatic, intuitive, anticipative intelligence of the people, which drawing upon the unfathomable sources of a rather intuitive than consciously recognized experience, not infrequently puts to shame the dialectically involved reasoning of scholasticism."

Paracelsus has indeed clearly expressed his opinion that theories should not be permitted to dominate the practice of the physician.

"For in experiments neither theories nor other arguments are applicable, but they are to be considered as their own expressions. Therefore we admonish every one who reads these, not to oppose the methods of experiment but according as its own power permits to follow it out without prejudice. For every experiment is like a weapon which must be used according to its peculiar power, as a spear to thrust, a club to strike,—so also is it with experiments.... To use experiments requires an experienced man who is sure of his thrust and stroke that he may use and direct it according to its fashion."[13]

That he endeavored to keep an open mind toward the

[13] *Chir. Bücher*, Fol. 1618, pp. 300-301.

symptoms of his patients, not too much governed by preconceived dogmas, is also indicated in his defense against certain attacks of his opponents in which they accuse him of not at once recognizing symptoms and treatment:

"They complain of me that when I come to a patient, I do not know instantly what the matter is with him, but that I need time to find out. It is indeed true that they pronounce judgment immediately; their folly is to blame for that, for in the end their first judgment is false, and from day to day as time passes they know less what the trouble is and hence betake themselves to lying, while I from day to day endeavor to arrive at the truth. For obscure diseases cannot be at once recognized as colors are. With colors we can see what is black, green, blue etc. If however there were a curtain in front of them we could not recognize them.... What the eyes can see can be judged quickly, but what is hidden from the eyes—it is vain to grasp as if it were visible. Take, for instance the miner; be he as able, experienced and skilful as may be, when he sees for the first time an ore, he cannot know what it contains, what it will yield, nor how it is to be treated, roasted, fused, ignited or burned. He must first run tests and trials and see whither these lead....Thus it is with obscure and serious diseases, that so hasty judgments cannot be made though the humoral physicians do this."[14]

Admitting the value of the positive contributions of Paracelsus to medical knowledge and practice, the net value of the reform campaign which he instituted is variously estimated by historians of medicine. For it must be remembered that Paracelsus fought against dogmas intrenched in tradition, by dogmas of his own. To the fantastic theories of the Greek-Arabian authorities he opposed many equally fantastic theories. That by his assault upon the absurdities and weaknesses of the Galenic medicine of

[14] *Op. fol.*, I, 262. (Die siebente Defension.)

UMI

his time he paved the way for greater hospitality to new and progressive ideas is unquestionable, but that by this assault he also did much to discredit the valuable elements as well as the corruptions of ancient medical achievements is also true. It is very difficult to balance justly the progressive and the reactionary influences he exerted upon the progress of medicine, and naturally, therefore, authorities differ upon this question. Thus Neuburger (*op. cit.*) appreciates the value of the accomplishments of Paracelsus, yet doubts that he is to be considered as a reformer of medicine in the sense that was Vesalius or Paré, that is, he laid no foundation stones of importance and the real value of much of his thought required the later developments of modern scientific thought for its interpretation. His aim was to found medicine upon physiological and biological foundation but the method he chose was not the right method, and his analogical reasons and fantastic philosophy of macrocosm and microcosm were not convincing and led nowhere. The disaffection and discontent with conditions in medicine produced by his campaign, can, thinks Neuburger, hardly be called a revolution. That was to come later through the constructive work of more scientific methods.

In a similar vein Haeser (*op. cit.*) remarks "Scarcely ever has a physician seized the problem of his life with purer enthusiasm, served it with truer heart, or with greater earnestness kept in view the honor of his calling than the reformer of Einsiedeln. But the aim of his scientific endeavors was a mistaken one and no less mistaken was the method by which he sought to attain it."

A recent writer, Professor Hugo Magnus,[15] presents a more critical point of view:

"We must then summarize our judgment to this effect, that Paracelsus keenly felt the frightful corruption which

[15] Hugo Magnus, *Paracelsus der Ueberarzt.* Breslau, 1906.

medicine and the investigation of nature suffered from the hands of the Scholastics, but that he did not understand how to penetrate to the causes of this condition of his science. Instead of seeking in the scholastic system the root of this medical degeneration, he believed that it must be found exclusively in the healing art of the ancients. And thus he sought to shatter in blind hatred all that existed, without being in position to be able to replace the old theory he maligned by a new and better concept of nature and medicine. So Paracelsus wore away in unclear struggling, his bodily and mental energy, and lived indeed as a reformer,—a medical superman, in his own imagination, in his own valuation, but not in the recognition of his own times, nor in the judgment of posterity."

"If therefore I can find no relationship between the general methods of medicine to-day and the Theophrastic concept of nature, nevertheless our supercolleague must be considered in an essentially limited respect, to be sure, as the pioneer in certain modern points of view. He was the first to attempt the consideration of the phenomena of organic life in a chemical sense, and I do not need to emphasize that he thereby paved the way to a very powerful advance in our science. In this respect was Paracelsus a reformer, here he has pointed new paths in the valuation of pathologic phenomena as well as in therapy, even if here also he has theorized enough and allowed his neo-Platonism to play him many a trick."

By discarding and condemning all the ancient authorities, thinks Magnus, Paracelsus assailed not only the corrupted Galenism of his time but did much to discredit the positive achievements of the Greeks, and although the original Greek authorities were not the then prevailing texts, they were at least accessible in newly translated versions, and the good in them might have been incorporated and built upon by Paracelsus if he had possessed the scientific

UMI

point of view. To the extent of his influence in this direction Paracelsus was therefore an opponent rather than a promoter of the progress of medical science. "Through his irrational theories he gave impulse to all sorts of mistaken notions among his followers, so that the wildest vagaries existed among the Paracelsists of the succeeding century."

The above will serve to illustrate the trend of modern critical judgment of Paracelsus as a reformer of medicine.

However estimates may vary as to the extent of the influence of Paracelsus as a reformer of medicine, credit must certainly be given him as a forceful agent in the downfall of the scholastic medical science of his time. The real reform in medical science, its establishment upon a basis of modern scientific method, was not the work of his century nor of the century to follow. Indeed it may not be too much to say that that great reform was mainly the work of the nineteenth century, and was made possible only through the patient labors of many investigators in the domains of physics, chemistry, anatomy, and biology.

If, however, we cannot claim for Paracelsus the unchallenged place of the reformer of medicine, we may at least recognize in him an earnest, powerful, and prophetic voice crying in the wilderness.

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THE ORIGIN OF THE MUTATION THEORY.

AT the time when Darwin published his book on the *Origin of Species* biological science was in a very different condition from what it is now. Hardly ten years had elapsed since Schleiden and Schwann discovered the fundamental law that all living organisms are built up of one or more ordinarily almost innumerable cells.

Mohl's contention that protoplasm is the essential and in fact the only living part of the cell is almost contemporaneous with Darwin's book (1849 and 1851). The presence of a nucleus within the cells began to be recognized. Hereditary problems were almost only discussed by breeders.

The *Textbook of Botany* by Julius Sachs appeared in 1868; it was the first to introduce into botany really scientific methods. When I was a student at the University of Leiden (1866-1870) systematic and descriptive morphological studies prevailed. Microscopical study of tissues was new and cytology had hardly reached us. Under these conditions a student interested in the causal relations of the phenomena of life naturally turned his mind to physics and chemistry. The prominent question of those days was the validity of physical and chemical laws in the living body. The idea dawned upon us that this question chiefly related to the protoplasm but hardly needed a proof for the cell walls and the tissues built up of them.

Once convinced that the phenomena of life are regu-

UMI

lated by the protoplasm we naturally looked for methods of studying this relation. Many different ways presented themselves, and among these four seemed to me the most promising. They were the study of respiration, of galls, of osmosis and of variability. I tried all of them and at the end chose the last. Respiration was the source of energy; it was a phenomenon common to animals and plants, and one of the main links which connected both kingdoms in our knowledge at that time. I devoted many years to its study, chiefly in a comparative way, and chose it for the subject of my inaugural address when I was called to the chair of plant physiology in the University of Amsterdam (1878).

But galls seemed to promise far more. They are built up of the ordinary qualities of the plants combined in a new way to fit the requirements of their insects, and this combination is brought about under the influence of some stimulus given off by the insect. To discover the nature of these stimuli and the laws by which they so effectively change the growth of the tissues, seemed to me a scope worth the devotion of a whole life. I made a large collection of galls, in search of the species which would be the most appropriate to attack this line of research. I concluded for those of the willows, belonging to the genus Nematus. But at that period I met with Mr. M. W. Beyerinck who was far beyond me in the study of the life history of the galls, and so I left this pathway. I have, however, read a course upon galls and their bearing on the broad problems of biology about every third year from that time on.

The study of osmosis and of the turgidity of the cells led to the discovery of the semi-permeable membranes of the protoplasm and their significance for growth and movements as well as for the study of isotonic coefficients and the determination of atomic weights, as, e. g., in the case

of the sugar raffinose. But its promise of elucidating hereditary questions diminished with every new discovery.

In 1880 I started a course on variability. I had been interested in this question chiefly by making a herbarium of monstrosities, and monstrosities were at that time almost all we knew of variability. Moreover I had visited the celebrated agriculturist W. A. Rimpau at Schlanstedt in Saxony and stayed repeatedly for some weeks on his estate in order to study his selection of cereals and sugarbeets. This induced me to take up a thorough study of agricultural and horticultural selection and I soon found that Darwin's books were the best guides for this literature. Especially from the pamphlets of Vilmorin, Verlot and Carrière I took a large part of the facts for elaboration of my lessons.

I read this course every second year from 1880 to 1900, and each time introduced into it the principles and methods which I found in the literature. This consisted partly in rare pamphlets which I succeeded in collecting only gradually, partly in articles scattered in agricultural and horticultural journals. In the meantime I increased my collection of monstrosities but soon perceived that collecting is not the right way to gain an insight into them. Therefore I preferred revisiting the same spots in nature for successive years and found the monstrosities regularly repeated. This induced the idea of their being heritable phenomena, a conception wholly new at that time, although the inheritance of the cockscomb or *Celosia* was, of course, known to every horticulturist. Then I turned to cultivation, made races of fasciated and twisted forms and studied the inheritance of pitchers and analogous deviations.

Parallel to these experimental studies I tried to penetrate into the theoretical side of the question, and this led to the publication of my book on *Intracellular Pangenesis* in 1889, of which the Open Court Publishing Company

UMI

published an English translation by Prof. C. Stuart Gager in 1910. Freed from the hypothesis of the transportation of germs through the tissues, Darwin's pangenesis coincided with my own conception of the material basis of protoplasmic life and of the hereditary qualities. This study brought about the conviction that variability must at least consist in two essentially different principles. One of them is the origin of new qualities and their accumulation through geological times, producing the continuous development of higher forms from lower. This form is what we now call mutability. The other is our present fluctuating variability. It determines the degree in which the single qualities will show in different individuals. I proposed this difference between mutability and fluctuating variability at the conclusion of my book, but said to myself: It is all right to deduce the theoretical necessity of this conclusion, but it would be of far higher importance to prove the actual existence of these two types of variation.

I set at work at once, first in the field but soon in the garden. I cultivated over a hundred wild species, and some of them through many years. Fluctuating variability was everywhere present. Then I chanced to meet with Quetelet's *Anthropométrie,* which had appeared in 1870, applied his methods to plants and saw that here the same general laws prevail. Different forms of curves of variation were determined in the corn marigold (*Chrysanthemum segetum*) and other plants (1894-1899), and it became clear that they changed the properties only in the directions of more or less development, but gave no indication whatever of an origin of new qualities. Fluctuation and mutability must therefore be principally distinct.

Mutations must of course be rare, but some few of them occurred in my garden in well-guarded breeds. They were sudden, without visible preparation or transitions. The peloric toadflax appeared in 1894, the double corn marigold

in 1896; they sufficed to prove the reality of mutations and gave an experimental basis for the appreciation and the study of the sudden appearance of new varieties in horticulture.

Besides them, one species proved to be rich in such sudden changes. It was Lamarck's evening primrose, a species originally wild in the eastern United States and collected there by Michaux, but which has since disappeared in America. It has, however, won an extensive distribution in England, Holland, Belgium and France, preferring the sand dunes along the coast. I observed its mutations for the first time in 1888 and since then it has never ceased to produce them. The number of mutants amounts to more than a dozen, some of them being progressive, as for instance the giant type or *Oenothera Lamarckiana gigas*, published in 1900, others retrogressive like the dwarfs and a brittle race called *O. rubrinervis*. Ordinarily they are constant from seed, but some show a splitting and are therefore considered to be half-mutants only, as *O. lata* and allied forms. The changes are always sudden and without transitions and occur so regularly in about 1% of the individuals that they constitute an unexpected but excellent material for experimental researches.

In my course on variability I laid especial stress on the pedigrees of definite systematic groups. The families of the euphorbiaceous and the umbelliferous plants afforded a very demonstrative material, and the hypothesis of the descent of the Monocotyls from the Dicotyls through types allied with the common buttercups, proposed at that time by Delpino, proved to be very convincing and instructive. Systematic atavisms, as shown in the leaf-bearing seedlings of the leafless species of Acacia and analogous instances were added to these discussions. They showed that evolution in nature is partly progressive and partly retrogressive. Progression means differentiation and speciali-

UMI

zation, it governs the main lines of the pedigree of the animal and vegetable kingdoms. But retrogression, consisting in the loss of previously developed qualities, must be responsible for a large part of the diversity of forms in nature. And since it is easier to lose a thing than to acquire a new quality, the cases of retrogression must be far more numerous in nature than those of actual progression.

Therefore there must be two kinds of mutations and even in our experimental cultures progressive ones must be rare, and retrogressive ones comparatively more frequent. This is exactly what we see in the mutations of the evening primrose.

Alongside of these studies I tried hybridization. Opium poppies afforded a useful material and led to the rediscovery of Mendel's law. At that time this conception was believed in by nobody, it was rather considered as an idealistic fiction. But the splitting of the poppies confirmed that of Mendel's peas, and numerous garden varieties behaved in the same way. I was fortunate enough to be the first to publish this result (1900) and pointed out that it is especially retrogressive variations which follow this law, whereas progressive ones produce constant hybrids, at least in many instances.

Paleontological studies strengthened the idea of the origin of species by means of sudden variations instead of a slow and gradual development. This side of the question has since been taken up by Charles A. White and other paleontologists. From my own studies I deduced the contention, that life on this earth has not lasted long enough for such a slow development as Darwin's theory of selection supposed. Darwin calculated some thousands of millions of years as required for his theory, but geologists and physicists only allow about forty or at most a hundred millions of years for the development of all animals and plants. The hypothesis of sudden mutations delivers us

from this difficulty. And so it does for many other objections which were still being used as weapons against the whole principle of evolution in the form proposed by Darwin.

It has always been my conviction that the improvement of industrial practice is the main aim of all science. Biological science has to be a basis for agriculture and horticulture. The discipline of heredity should be crowned by the advance in our knowledge concerning the breeding of animals and plants. With Dr. Wakker I studied the diseases of the flower bulbs cultivated all around Haarlem (1883-1885), and since then I regularly sent contributions to the journal of our agricultural society. From 1892 to 1894 I was editor of the journal of the Dutch Horticultural Society in order to have an easy access to horticultural establishments in the Netherlands as well as abroad, and collected all the evidence I could find concerning practical plant-breeding. As a matter of fact this was very scanty but it led me to a connection with the Director of the Swedish agricultural station at Svalöf, Dr. Hjalmar Nilsson, whose celebrated method of plant improvement rested on the same scientific basis as my own experiments.

My book on the mutation theory is the combination of all these preliminary studies into a regular discussion of the main principle. I had the great advantage of my steadily repeated courses on heredity, which constituted, if I may say so, a first unpublished edition, with all the many faults inherent to first trials on a new field. The book appeared in 1900, and an English edition,[1] prepared by Prof. J. B. Farmer and A. D. Darbishire, was published by the Open Court Publishing Company in 1909. It tries to show that the origin of species is a natural phenomenon and that it is possible to subject it to experimental study. In nature the mutations have produced the whole evolution

[1] *The Mutation Theory.* 2 vols.

UMI

of all living beings; in the garden we can, of course, only expect to see their very smallest steps. The identity of retrogressive mutations in nature, in horticulture and agriculture and in the experimental garden seems now to be beyond doubt. But progressive changes, which are the most important, are at the same time the rarest, in nature as well as in cultivation. In regard to these the theory relies on its broad arguments and the question whether the directly observed progressive mutations afford a material for the interpretation of the ways of nature is still under discussion.

The theory is based upon arguments taken from widely different branches of nearly all natural sciences. It conduces of necessity to experimental research, but this, of course, is still in its first infancy. It promises, however, to become some day of important service to science at large as well as to the practice of breeders.

HUGO DE VRIES.

LUNTEREN, HOLLAND.

THE MANUSCRIPTS OF LEIBNIZ ON HIS DISCOVERY OF THE DIFFERENTIAL CALCULUS.

PART II (CONTINUED).

§§ XI—XV.

Between the date of the manuscript last considered and the one which follows there is a gap of seven months, for which Gerhardt does not appear to have found anything. This is very unfortunate; for in this interval Leibniz has attained to the important conclusion that *the true general method of tangents is by means of differences.* We saw that in November 1675 he had *started* to investigate more thoroughly the direct method of tangents; but the method is that of the auxiliary curve, and there is no indication whatever of the characteristic triangle. Does this interval correspond with the time taken by Leibniz for his final reading of Barrow from Lect. VI to Lect. X, comparing all the geometrical theorems with his own notation? Or is it only a strange coincidence that Leibniz's order is the same as that of Barrow, first the auxiliary curve, and lastly the method of differences? One could form a more definite opinion, if Leibniz had given a diagram for the first problem he considers, the one in the next following manuscript, which amounts to the differentiation of an inverse sine. Such a diagram he must have had beside him as he wrote; for I think the reader will find that he wants one to follow the argument; with the idea

UMI

of verifying this argument, I have not endeavored to supply the omission.

The consideration of the direct method of tangents is apparently, however, only as a means and not as an end; for Leibniz harks back to the inverse method, and to the catalogue of quadrible curves, which he seems to say he has in hand. It is not until November 1676 that he seems to be coming into his own; and it is not until July 1677 that he has a really definite statement of his rules. On the other hand, in July 1676, he is consistently using the differential factor with all his integrals, and before the end of that year he has the differential of a product, whether obtained as the inverse of his theorem $\int y\,dx = xy - \int x\,dy$, or by the use of the substitution $x + dx$, $y + dy$, is not certain; but this substitution appears in the manuscript for November 1676. Finally, in July 1677, appears the general idea of the substitution of other letters, in order to eliminate the difficulty caused by the appearance of the variable under a root sign or in the denominator of a fraction; and with this the whole thing is now fairly complete for all *algebraical functions*. There is as yet no equally clear method for the treatment of exponentials, logarithms, or trigonometrical functions; for the latter he refers to a geometrical diagram, strongly reminiscent of Barrow.

§ XI.

26 June, 1676.

Nova methodus Tangentium.
(New Method of Tangents.)

I have many beautiful theorems *with regard to the method of tangents both direct as well as inverse.* Descartes's method of tangents depends on finding two equal roots, and it cannot be employed, except in the case when all the undetermined quantities occurring in the work are expressible in terms of one, for instance, in terms of the abscissa.

But the true general method of tangents is by means of dif-

ferences. That is to say, the difference of the ordinates, whether direct or converging, is required. It follows that quantities that are not amenable to any other kind of calculus are amenable to the calculus of tangents, so long as their differences are known. Thus if we are given an equation in three unknowns, in which x is an abscissa, y an ordinate, and z the arc of a circle of which x is the sine of the complement, e. g., the equation $b^2y = cx^2 + fz^2$. To find the next consecutive y, in place of x take $x+\beta$, and in place of z take $z-dz$, or, since $\overline{dz} = \frac{\beta r}{\sqrt{r^2-x^2}}$, we may take $z - \frac{\beta r}{\sqrt{r^2-x^2}}$ [51]; hence we have

$$b^2(y) = cx^2 + 2cx\beta + c\beta^2 + fz^2 - \frac{2fz\beta r}{\sqrt{r^2-x^2}} + \frac{\beta^2 r^2}{r^2-x^2}$$

Hence the difference between y and (y) is given by

$$\pm b^2y \mp b^2(y) = +2cx\beta - \frac{2fz\beta r}{\sqrt{r^2-x^2}} = b^2\,\overline{dy};$$

Therefore $$\frac{dy}{\beta} = \frac{\mp 2cx\sqrt{r^2-x^2} \mp 2fzr}{b^2\sqrt{r^2-x^2}} = \frac{t}{y} = \frac{tb^2}{cx^2+fz^2}.$$

From this the flexure or sinuosity of the curve can be found, according as now $2cz\sqrt{r^2-x^2}$, now $2fzr$ predominates; for when they are equal, the ordinate on that side on which it was previously the greater then becomes the less. It is just the same, if several other undetermined quantities, such as logarithms and other things occur, no matter how they are affected, as for instance in the equation $b^2y = cx^2 + fz^2 + xzl$, where z is supposed to be an arc, and l a logarithm, x the sine of the complement of the arc, and y the number of the logarithm, b being the radius and unity, equal to r. Also it is just the same, whenever an undetermined transcendental has been derived from some dimension or quadrature that has not been investigated.[52]

For the rest, many noteworthy and useful theorems now arise from the foregoing by the inverse method of tangents. Thus general equations, or equations of any indefinite degree may be formed, at first indeed in two unknowns, x and y, only. But if in this way the matter does not work out satisfactorily, it will easily do so when

[51] In this and the following line I have corrected two obvious misprints; they are evidently not the fault of Leibniz, for the lines that follow from them are correct.

[52] There is some doubt here as to whether Leibniz could have given an example; but it must be remembered that these are practically only notes, mostly for future consideration.

of all living beings; in the garden we can, of course, only expect to see their very smallest steps. The identity of retrogressive mutations in nature, in horticulture and agriculture and in the experimental garden seems now to be beyond doubt. But progressive changes, which are the most important, are at the same time the rarest, in nature as well as in cultivation. In regard to these the theory relies on its broad arguments and the question whether the directly observed progressive mutations afford a material for the interpretation of the ways of nature is still under discussion.

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THE MANUSCRIPTS OF LEIBNIZ ON HIS DISCOVERY OF THE DIFFERENTIAL CALCULUS.

PART II (CONTINUED).

§§ XI—XV.

Between the date of the manuscript last considered and the one which follows there is a gap of seven months, for which Gerhardt does not appear to have found anything. This is very unfortunate; for in this interval Leibniz has attained to the important conclusion that *the true general method of tangents is by means of differences.* We saw that in November 1675 he had *started* to investigate more thoroughly the direct method of tangents; but the method is that of the auxiliary curve, and there is no indication whatever of the characteristic triangle. Does this interval correspond with the time taken by Leibniz for his final reading of Barrow from Lect. VI to Lect. X, comparing all the geometrical theorems with his own notation? Or is it only a strange coincidence that Leibniz's order is the same as that of Barrow, first the auxiliary curve, and lastly the method of differences? One could form a more definite opinion, if Leibniz had given a diagram for the first problem he considers, the one in the next following manuscript, which amounts to the differentiation of an inverse sine. Such a diagram he must have had beside him as he wrote; for I think the reader will find that he wants one to follow the argument; with the idea

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of verifying this argument, I have not endeavored to supply the omission.

The consideration of the direct method of tangents is apparently, however, only as a means and not as an end; for Leibniz harks back to the inverse method, and to the catalogue of quadrible curves, which he seems to say he has in hand. It is not until November 1676 that he seems to be coming into his own; and it is not until July 1677 that he has a really definite statement of his rules. On the other hand, in July 1676, he is consistently using the differential factor with all his integrals, and before the end of that year he has the differential of a product, whether obtained as the inverse of his theorem $\int y\,dx = xy - \int x\,dy$, or by the use of the substitution $x + dx$, $y + dy$, is not certain; but this substitution appears in the manuscript for November 1676. Finally, in July 1677, appears the general idea of the substitution of other letters, in order to eliminate the difficulty caused by the appearance of the variable under a root sign or in the denominator of a fraction; and with this the whole thing is now fairly complete for all *algebraical functions*. There is as yet no equally clear method for the treatment of exponentials, logarithms, or trigonometrical functions; for the latter he refers to a geometrical diagram, strongly reminiscent of Barrow.

§ XI.

26 June, 1676.

Nova methodus Tangentium.
(New Method of Tangents.)

I have many beautiful theorems *with regard to the method of tangents both direct as well as inverse.* Descartes's method of tangents depends on finding two equal roots, and it cannot be employed, except in the case when all the undetermined quantities occurring in the work are expressible in terms of one, for instance, in terms of the abscissa.

But the true general method of tangents is by means of dif-

ferences. That is to say, the difference of the ordinates, whether direct or converging, is required. It follows that quantities that are not amenable to any other kind of calculus are amenable to the calculus of tangents, so long as their differences are known. Thus if we are given an equation in three unknowns, in which x is an abscissa, y an ordinate, and z the arc of a circle of which x is the sine of the complement, e. g., the equation $b^2y = cx^2 + fz^2$. To find the next consecutive y, in place of x take $x+\beta$, and in place of z take $z-dz$, or, since $\overline{dz} = \frac{\beta r}{\sqrt{r^2-x^2}}$, we may take $z - \frac{\beta r}{\sqrt{r^2-x^2}}$ [51]; hence we have

$$b^2\,(y) = cx^2 + 2cx\beta + c\beta^2 + fz^2 - \frac{2fz\beta r}{\sqrt{r^2-x^2}} + \frac{\beta^2 r^2}{r^2-x^2}$$

Hence the difference between y and (y) is given by

$$\pm b^2y \mp b^2(y) = +2cx\beta - \frac{2fz\beta r}{\sqrt{r^2-x^2}} = b^2\,\overline{dy};$$

Therefore $$\frac{dy}{\beta} = \frac{\mp 2cx\sqrt{r^2-x^2} \mp 2fzr}{b^2\sqrt{r^2-x^2}} = \frac{t}{y} = \frac{tb^2}{cx^2+fz^2}.$$

From this the flexure or sinuosity of the curve can be found, according as now $2cz\sqrt{r^2-x^2}$, now $2fzr$ predominates; for when they are equal, the ordinate on that side on which it was previously the greater then becomes the less. It is just the same, if several other undetermined quantities, such as logarithms and other things occur, no matter how they are affected, as for instance in the equation $b^2y = cx^2 + fz^2 + xzl$, where z is supposed to be an arc, and l a logarithm, x the sine of the complement of the arc, and y the number of the logarithm, b being the radius and unity, equal to r. Also it is just the same, whenever an undetermined transcendental has been derived from some dimension or quadrature that has not been investigated.[52]

For the rest, many noteworthy and useful theorems now arise from the foregoing by the inverse method of tangents. Thus general equations, or equations of any indefinite degree may be formed, at first indeed in two unknowns, x and y, only. But if in this way the matter does not work out satisfactorily, it will easily do so when

[51] In this and the following line I have corrected two obvious misprints; they are evidently not the fault of Leibniz, for the lines that follow from them are correct.

[52] There is some doubt here as to whether Leibniz could have given an example; but it must be remembered that these are practically only notes, mostly for future consideration.

the tables which I am investigating are finished; then it will be possible to take one or more other letters, and to take the difference as an arbitrary known formula, and when this is done it is certain that finally in any case a formula will be found such as is required, and in this way also a curve which will satisfy the conditions given; but in truth the description of the curve will need diagrams for these symbols, representing the sums of the arbitrarily chosen differences.

Now once a curve is found having the tangent property that we want, it will be more easy afterwards to find simpler constructions for it. We have this also as a convenient means enabling us to use many quantities that are transcendent, yet depending the one on the other, such for example as are all those that depend on the quadrature of the circle or the hyperbola. From these investigations it will also appear whether or no other quadratures can be reduced to the quadrature of the circle or the hyperbola. Lastly, since the finding of maxima and minima is useful for the inscription and circumscription of polygons, hence also, by employing these transcendent magnitudes, convergent series can be found, and in the same way their terminations; or of any quantities formed in the same way. However in that case it may not be so easy to argue about impossibility; at least indeed by the same method. Only I do not see how we can find whether from the quadrature of the circle, say, any sum can be found, when no quantity depending on the dimensions of the circle enters into the calculation.

§ XII.

July, 1676.

Methodus tangentium inversa.

[Inverse method of tangents.]

In the third volume of the correspondence of Descartes, I see that he believed that Fermat's method of Maxima and Minima is not universal; for he thinks (page 362, letter 63) that it will not serve to find the tangent to a curve, of which the property is that the lines drawn from any point on it to four given points are together equal to a given straight line.

[Thus far in Latin; Leibniz then proceeds in French.]

Mons. des Cartes (letter 73, part 3, p. 409) to Mons. de Beaune.

"I do not believe that it is in general possible to find the converse to my rule of tangents, nor of that which Mons. Fermat uses,

although in many cases the application of his is more easy than mine; but one may deduce from it *a posteriori* theorems that apply to all curved lines that are expressed by an equation, in which one of the quantities, x or y, has no more than two dimensions, even if the other had a thousand. There is indeed another method that is more general and *a priori*, namely, by the intersection of two tangents, which should always intersect between the two points at which they touch the curve, as near one another as you can imagine; for in considering what the curve ought to be, in order that this intersection may occur between the two points, and not on this side or on that, the construction for it may be found. But there are so many different ways, and I have practised them so little, that I should not know how to give a fair account of them."

Mons. des Cartes speaks with a little too much presumption about posterity; he says (page 449, letter 77) that his rule for resolving in general all problems on solids has been without comparison the most difficult to find of all things which have been discovered in geometry up to the present, and one which will possibly remain so after centuries, "unless I take upon myself the trouble of finding others" (as if several centuries would not be capable of producing a man able to do something that would be of greater moment).

(Page 459.) The question of the four spheres is one that is easy to investigate for a man who knows the calculus. It is due to Descartes, but as it is given in the book, it appears to be very prolix.

The problem on the inverse method of tangents, which Mons. des Cartes says he has solved (Vol. 3, letter 79, p. 460)

[Leibniz then continues in Latin.]

EAD is an angle of 45 degrees. ABO is a curve, BL a tangent to it; and BC, the ordinate, is to CL as N is to BJ. Then

$$\mathrm{CL}=\frac{\mathrm{BC}=ny}{\mathrm{BJ}=y-x},\quad \mathrm{CL}=t,$$

hence, $$t=\frac{ny}{y-x},\quad \frac{n}{t}=\frac{y-x}{y}=1-\frac{x}{y},$$

hence, $$\frac{x}{y}=\frac{t-n}{t};\ \text{but}\ \frac{t}{y}=\frac{\overline{dx}}{dy};$$

therefore $$\frac{\overline{dx}}{dy}=\frac{n}{y-x},\ \text{or}\ \overline{dx}\,y-x\,\overline{dx}=\overline{dy}\,n;$$

hence $$\int \overline{dx}\,y-\int x\,\overline{dx}=-n\int dy.$$

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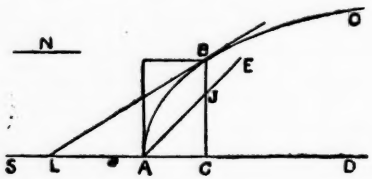
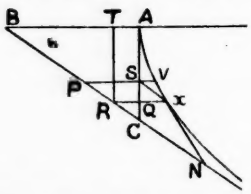
Now, $\int \overline{dy} = y$, and $\int \overline{x\,dx} = x^2/2$, and $\int \overline{dx\,y}$ is equal to the area ACBA, and the curve is sought in which the area ACBA is equal to $(x^2/2) + ny = (AC^2/2) + n\text{BC}$. [53]



Let this $x^2/2$, i. e., the triangle ACJ be cut off from the area, then the remainder AJBA should be equal to the rectangle ny.

The line that de Beaune proposed to Descartes for investigation reduces to this, that if BC is an asymptote to the curve, BA the axis, A the vertex, AB, BC, fixed lines, for BAC is at right angles.



Let RX be an ordinate, XN a tangent, then RN is always to be constant and equal to BC; required the nature of the curve.

This is how I think it should be done.

Let PV be another ordinate, differing from the other one RX by a straight line VS, found by drawing XS parallel to RN; then

[53] Leibniz has a footnote to this manuscript: "I solved in one day two problems on the inverse methods of tangents, one of which Descartes alone solved, and the other even he owned that he was unable to do."

This problem is one of them, the first mentioned in the footnote given by Leibniz. But it requires a stretch of imagination to consider Leibniz's result as a solution. For he ends up with a geometrical construction, that is at least as hard as the construction that can be made by the use of the original data. There is of course the usual misprint that one is becoming accustomed to; but there is also the unusual, for Leibniz, mistake of using his data incorrectly. Starting with the hypothesis that BC: CL = N: BJ, he writes CL = N.BC/BJ (correcting the omission of the factor N), instead of CL = BC.BJ/N.

The solution of the problem is $y + n\log(y - x + n) = 0$, as originally stated, or $x = n\log(n - y + x)$, if we continue from Leibniz's erroneous result $dx/dy = n/(y - x)$.

The point to be noted, however, is that Leibniz does not remark that "this curve appertains to a logarithm."

the triangles SVX, RXN are similar, RN $= t = c$, a constant, RX $= y$, SY $= dy$, and therefore

$$\frac{\overline{dy}}{dx} = \frac{y}{t=c} \text{ ; hence } cy = \int \overline{y\,dx} \quad \text{or } c\,\overline{dy} = y\,\overline{dx}. \quad [54]$$

If AQ or TR $= z$, and AC $= f$, while BC $= a$;

then, $$\frac{\text{AC}}{\text{BC}} = \frac{f}{a} = \frac{\text{TR}}{\text{BR}} = \frac{z}{x}; \text{ and thus } x = \frac{az}{f}.$$

If $\overline{dx}$ is constant, then $\overline{dz}$ is also constant. Hence

$c\,dy = \frac{a}{f} y\,\overline{dz}$, or $cy = \frac{a}{f}\int \overline{y\,dz}$, and $cy\,\overline{dy} = \frac{a}{f} y^2\,\overline{dz}$, therefore

$c\frac{y^2}{2} = \frac{a}{f}\int y^2\,\overline{dz}$. Hence we have both the area of the figure and the moment to a certain extent (for something must be added on account of the obliquity); also

$cz\,\overline{dy} = \frac{a}{f} yz\,\overline{dz}$, and therefore $c\int z\,\overline{dy} = \frac{a}{f}\int \overline{yz\,dz}$.

Also $\frac{c\overline{dy}}{y} = \frac{a}{f}dz$, and hence, $c\int \frac{\overline{dy}}{y} = \frac{a}{f}z$. Now, unless I am greatly mistaken, $\int \frac{\overline{dy}}{y}$ is in our power.[55] The whole matter reduces to this, we must find the curve[56] in which the ordinate is such that

[54] Leibniz does not see that this result immediately gives him the equation that he requires. Thus $x = c\,\text{Log}\,y$, as he would have written it; the usual omission of the arbitrary constant does not matter in this case, so long as BA is taken as unity, which is possible with Leibniz's data.

[55] Here he seems to recognize that he has the solution. The next sentence is, however, very strange. As long ago as Nov. 1675 he has written $\int a^2/y$ as Log y, and recognized the connection between the integral and the quadrature of the hyperbola; and yet he says "unless I am mistaken, $\int dy/y$ is always in our power." Now notice that in the date there is no day of the month given, contrary to the usual custom with these manuscripts so far; can it be possible that this date was afterward added from memory, and that the manuscript should bear an earlier date? If not we must conclude that Leibniz has not yet attained to a correct idea of the meaning of his integral sign, and is still worried by the necessity (as it appears to him) of taking the y's in arithmetical progression.

[56] The passage in the original Latin is very ambiguous, and it may be that it is not quite correctly given; I think, however, that I have given the correct idea of what Leibniz intended. One has to draw an *auxiliary* curve, in which $y = dy/dx$, and then find its area; in that case it should be "divided by the differences of the abscissae" instead of "divided by the abscissae."

it is equal to the differences of the ordinates divided by the abscissae, and then find the quadrature of that figure.

$$d\sqrt{ay} = \frac{1}{\sqrt{ay}} \qquad (57)$$

Figures of this kind, in which the ordinates are dy/y, dy/y^2 dy/y^3, are to be sought in the same way as I have obtained those whose ordinates are $y\,dy$, y^2dy, etc. Now $w/a = \overline{dy}/y$, and since $\overline{dy}$ may be taken to be constant and equal to β,[58] therefore the curve, in which $w/a = \overline{dy}/y$, will give $wy = a\beta$, which would be a hyperbola.[59] Hence the figure, in which $dy/y = z$, is a hyperbola, no matter how you express y, and if y is expressed by ϕ^2 we have $dy = 2\phi$, and $\frac{2\phi}{\phi^2} = \frac{2}{\phi}$. Now, $c\int\frac{dy}{y} = \frac{a}{f}z$, and therefore $\frac{fc}{a}\int\frac{1}{y} = z$, which thus appertains to a logarithm.[60]

Thus we have solved all the problems on the inverse method of tangents,[61] which occur in Vol. 3 of the Correspondence of Descartes, of which he solved one himself, as he says on page 460, letter 79, Vol. 3; but the solution is not given; the other he tried to solve but could not, stating that it was an irregular line, which in any case was not in human power, nay not within the power of the angels unless the art of describing it is determined by some other means.

§ XIII.

This manuscript bears no date: however, it was probably written very shortly after his call on Hudde at Amsterdam, on his way home from England (the second visit)

[57] An interpolated note, marking a sudden thought or guess; for the next sentence carries on the train of thought that has gone before. Query, some interval of time, either short (such as for a meal) or long (continued the next day), may have occurred here.

[58] This cannot be referred back to the present problem, since Leibniz has already assumed in it that dz and dx are constant. This may account for the fact that he has hesitated to say that the integral represents a logarithm.

[59] This working is intended to apply to the auxiliary curve mentioned above, w standing for dx, and β for dy; hence the curve is not a hyperbola; Leibniz seems to have been misled by the appearance of the equation suggesting $xy =$ constant.

[60] Here apparently he leaves the muddle, in which he has entangled himself, and returns to his original equation; he then remembers that he has found before that the integral in question leads to a logarithm.

[61] He has not solved either of them; nor can it be said from this that "Leibniz in 1676 sought and found the curve whose subtangent is constant." Of all the work that Leibniz has done hitherto, there is none that is so inconclusive as this in comparison.

to Hanover. Leibniz stayed in Holland from October 1676 to December of that year; hence the date may be fairly accurately assigned.

Hudde showed me that in the year 1662 he already had the quadrature of the hyperbola, which I found was the very same as Mercator also had discovered independently, and published. He showed me a letter written to a certain van Duck, of Leyden I think, on this subject. His method of tangents is more complete than that of Sluse, in that he is able to use any arithmetical progression, as in a simple equation, whereas Sluse and others can use only one. Hence constructions can be made simple, while terms can be eliminated at will. This also can be made use of for eliminating any letter with greater facility, for numerous equations of all sort are thereby rendered fit for elimination.

$$\begin{array}{l} x^3 + px^2 + qx = 0 \\ \quad y \quad y, y \\ \quad\quad y^2\ y^2 \\ \quad\quad\quad y^3 \\ \hline 3x^2 + 2px^2 + qx\ 0 \\ \quad 2yx^2 + yx \\ \quad\quad y^2x \end{array} \qquad \begin{array}{l} x^2 + xy + y^2 + x + y + a = 0 \\ 2x\overline{dx} + x\overline{dy} + 2y\overline{dy} + \overline{dx} + \overline{dy} = 0 \\ \quad\quad y\overline{dx} \\ \\ \frac{t}{y} = \frac{\overline{dx}}{\overline{dy}} = \frac{x+2y+1}{y+2x+1} \end{array}$$

What I had observed with regard to triangular numbers for three equal roots, and pyramidal numbers for four, was already known to him, and indeed even more generally,

		−1	0	1	2	3	4	5	6
−3	−1		0	0	1	3	6	10	15
−4	−1	0		0	0	1	4	10	20

Here it must be observed that the number of zeros increases, as this is of the greatest service in separating roots.

He has also rules for multiplying equations, so that they are not only determined for equal roots, but also for roots increasing arithmetically, or geometrically, or according to any progression.

Hudde has a most elegant construction for describing two curves, one outside and the other inside a circle, which are capable of quadrature, and by means of these curves he finds the true area of a circle so nearly, that with the help of the dodecagon, in a number of six figures, there is an error of only three units, or 3/100000.

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He has a method for finding the real roots of equations, having some roots real and the rest impossible, by the help of another equation having all its roots real, and as many in number as he previously had of real and impossible together.

He had an example of a beautiful method of finding sums of series by the continuous subtractions of geometrical progressions. He subtracts geometrical progressions whose sums are also geometrical progressions, and thus he can find the sums of the sums, and so he obtains the sum of the series. This method is excellent for a series whose numerators are arithmetical, and denominators geometrical, such as,

$$\frac{1}{2}\ \frac{2}{4}\ \frac{3}{8}\ \frac{4}{16}\ \ldots\ldots\ .$$

He has three series, like those of Wallis, for interpolations for the circle. He says that there are no more by that method, I think.

Also he can very often write down the quadratures of irrationals, as also their tangents, without eliminating irrationals, or fractions, etc.

§ XIV.

November, 1676.

Calculus Tangentium differentialis.
[Differential calculus of tangents.]

$$\overline{dx} = 1,\quad \overline{dx^2} = 2x,\quad \overline{dx^3} = 3x^2,\quad \text{etc.}$$

$$\overline{d\frac{1}{x}} = -\frac{1}{x^2},\quad \overline{d\frac{1}{x^2}} = -\frac{2}{x^2},\quad d\frac{1}{x^3} = \frac{3}{x^2},\quad \text{etc.}$$

$$\overline{d\sqrt{x}} = \frac{1}{\sqrt{x}},\quad \text{etc.}$$

From these the following general rules may be derived for the differences and sums of the simple powers:

$$\overline{dx^e} = e, x^{e-1},\ \text{and conversely}\ \int\overline{x^e} = \frac{x^{e+1}}{e+1}.$$

Hence, $\overline{d\frac{1}{x^2}} = \overline{dx^{-2}}$ will be $-2x^{-3}$ or $-\frac{2}{x^3}$,

and $\overline{d\sqrt{x}}$ or $dx^{\frac{1}{2}}$ will be $-\frac{1}{2}x^{-\frac{1}{2}}$ or $-\frac{1}{2}\sqrt{\frac{1}{x}}$.

Let $y = x^2$, then $\overline{dy} = 2x\ \overline{dx}$ or $\frac{\overline{dy}}{dx} = 2x$.

This reasoning is general, and it does not depend on what the progression for the x's may be.[62] By the same method, the general rule is established as:

$$\frac{\overline{dx^e}}{dx} = e\,x^{e-1}, \text{ and } \int \overline{x^e\,dx} = \frac{x^{e+1}}{e+1}.$$

Suppose that we have any equation whatever, say,

$$ay^2 + byx + cz^2 + f^2x + g^2y + h^3 = 0,$$

and suppose that we write $y+dy$ for y, and $x+dx$ for x, we have, by omitting those things which should be omitted, another equation

$$\left.\begin{array}{l} ay^2 + byx + cx^2 + f^2x + g^2y + h^3 = 0 \\ \hline a2\overline{dy}y + by\overline{dx} + 2cx\overline{dx} + f^2dx + g^2\overline{dy} \\ \qquad\quad bx\overline{dy} \\ \hline a\,\overline{dy}^2 + b\,\overline{dx}\overline{dy} + c\,dx^2 = 0 \end{array}\right\} = 0 \qquad (63)$$

This is the origin of the rule published by Sluse. It can be extended indefinitely: Let there be any number of letters, and any formula composed from them; for example, let there be the formula made up of three letters,

$$ay^2 \quad bx^2 \quad cz^2 \quad fyx \quad gyx \quad hxz \quad ly \quad mx \quad nz \quad p = 0.$$

From this we get another equation

$$\begin{array}{cccccccc} ay^2 & bx^2 & cz^2 & fyx & \text{simi-} & ly & mx & \text{simi-}\ p \\ \hline 2a\overline{dy}y & 2b\overline{dx}x & 2c\overline{dz}z & fy\overline{dx} & \text{larly} & l\overline{dy} & m\overline{dx} & \text{larly} \\ & & & fx\overline{dy} & & & & \\ \hline a\,\overline{dy}^2 & b\overline{dx}^2 & c\overline{dz}^2 & f\overline{dx}\overline{dy} & \ldots\ldots & & & \end{array}$$

It is plain from this that by the same method tangent planes

[62] AT LAST! The recognition of the fact that neither dx nor dy need necessarily be constant, and the use of another letter to stand for the function that is being differentiated, mark the beginning, the true beginning, of Leibniz's development of differentiation. Later in this manuscript we find him using the third great idea, probably suggested by the second of those given above, namely, the idea of substitution, by means of which he finally attains to the differentiation of a quotient, and a root of a function.

It is very suggestive that this remarkable advance occurs after his second visit to London, while he is staying in Holland. Did some one tell then of the work of Newton, or of Barrow's method (which is geometrically an exact equivalent of substitution), pointing out those things of which he had not perceived the drift, or is it the result of his intercourse with Hudde? For the date is that of his stay at The Hague. (For the answer to this query see an article to follow, entitled "Leibniz in London."—Ed.)

[63] This is Barrow all over; even to the words *omissis omittendis* instead of Barrow's *rejectis rejiciendis*. Lect. X, Ex. 1 on the differential triangle at the end of the lecture.

to surfaces may be obtained, and in every case that it does not matter whether or no the letters x, y, z have any known relation, for this can be substituted afterward.

Further, the same method will serve admirably, even though compound fractions or irrationals enter into the calculation, nor is there any need that other equations of a higher degree should be obtained for the purpose of getting rid of them; for their differences are far better found separately and then substituted; hence the ordinary method of tangents will not only proceed when the ordinates are parallel, but it can also be applied to tangents and anything else, aye, even to those things that are related to them, such as proportions of ordinates to curves, or where the angle of the ordinates changes according to some determined law. It will be worth while especially to apply the method to irrationals and compound fractions.[64]

$$\overline{d\sqrt[2]{a+bz+cz^2}}. \quad \text{Let } a+bz+cz^2=x;$$

then $$\overline{d\sqrt[2]{x}}=-\frac{1}{2\sqrt{x}}, \text{ and } \frac{dx}{dz}=b+2cz;$$

therefore $$\overline{d\sqrt[2]{a+bz+cz^2}}=-\frac{b+2cz}{2\,dz\,\sqrt{a+bz+cz^2}}.$$

Taking any equation between two letters x and y for a curve, and determining the equation of the tangent, either of the two letters x or y can be eliminated, so that all that remains is the other together with $\overline{dx}$ and $\overline{dy}$; and this will be worth while doing in all cases to facilitate the calculation.

If three letters are given, say x, y and z, and the value of $\overline{dz}$ is expressed in terms of x or y (or even of both), an equation for the tangents will at length be obtained, in which again there will be left only one or other of the letters x or y together with the two, $\overline{dx}$ and $\overline{dy}$; sometimes z itself cannot be eliminated. Also this can be deduced in all cases of an assumed value of $\overline{dz}$, and in the same way more additional letters can be taken. Thus, bringing together every general calculus into one, we obtain the most general of them all. Besides, the assumption of a large number of letters may be employed to solve problems on the inverse method of tangents, with the assistance of quadratures.

[64] Here we have the idea of substitutions, which made the Leibnizian calculus so superior to anything that had gone before. Note that he still has the erroneous sign that he obtained for the differentiation of $\sqrt{x}$ at the beginning of this manuscript. Also that the dz is wrongly placed in the denominator of the result.

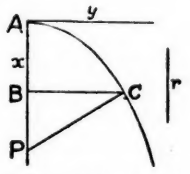
Thus, if the following problem is set for solution: It is given that the sum of the straight lines CB, BP or

$$y + y\frac{\overline{dy}}{dx} = xy;$$

we have

$$\overline{dx} + \overline{dy} = x\overline{dx}$$

$$\text{or } x + y = \frac{x^2}{2}.$$



Thus we have the curve in which the sum of CB + BP (multiplied by a constant r) is equal to the rectangle AB.BC.

There are two marginal notes by Leibniz that must be referred to, in this manuscript. The first reads:

It is especially to be observed about my calculus of differences that, if

$$b, ydx + xdy + \text{etc.} = 0$$

then $byx + f$ etc. $= 0$, and so on for the rest. It is to be seen what is to be done about the h^3. For the purpose of making these calculations better, the equation $ay^2 + byx + cx^2 +$ etc. can be changed into something else by means of another relation of the curve, and if it turns out all right it may be compared to another calculation of the differences, since it comes to the thing as by the first. The two points to be noticed are that Leibniz now for the first time recognizes the need of considering the arbitrary constant of integration, though he hardly grasps how it arises, and that even now he cannot refrain from harking back to his obsession of the obtaining of several equations for comparison. This note is not made any the easier to understand by its being starred by Gerhardt for reference to the differentiation of x^2, whereas it obviously (when you come later to the passage) refers to the differentiation of the equation of the second degree.

The second note refers to the substitution of $x + dx$ for x and $y + dy$ for y, and reads:

Either dx or dy can be expressed arbitrarily, a new equation being obtained; and either dx or dy being taken away, x, or y, say, can be otherwise expressed in terms of the quantities. It is not true, I think, that this is so, for then a catalogue of all curves capable of quadrature would result, by supposing one or other of them to be constant.

The point to be noticed in this rather ambiguous statement is that Leibniz is still thinking of his catalogue, and is not himself convinced of the completeness of his method for all purposes.

§ XV.

There is an interval of nearly seven months between the date of the manuscript last considered and the one that now follows. This interval has been full of work; for we now find a clear exposition of the rules for the differentia-

tion of a sum, difference, product, quotient, etc., though these are without proof, or indication of the manner in which they have been obtained. There is also no rule given for a logarithm, an exponential, or a trigonometrical ratio. Leibniz may have known them, but even then it would not be surprising to find them left out; for Leibniz's great idea was the use of his method to facilitate calculation. We must conclude therefore that these rules are a development of the method of substitution outlined in the preceding manuscript.

This essay has several peculiar characteristics of its own, which distinguish it from those that have gone before. It is written throughout in French; it is to some extent historical and critical, having the appearance of being prepared for publication, or possibly as a letter; this is corroborated by the fact that there is an original draft and a more fully detailed revision. Could it be that this is the original of Leibniz's communication of this method to Newton and others? If so, Leibniz is very careful not to give much away. The figures are strongly reminiscent of Barrow, but the context does not deal with subtangents, which are such a feature in all Barrow's work.

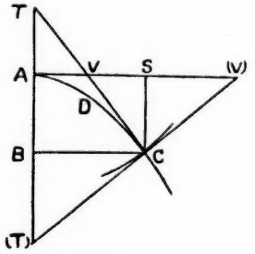
The start from the work of Sluse is peculiar; it seems to suggest that Leibniz is pointing out that his method is a fuller development of that of the former. Leibniz has already hazarded two different guesses at the origin of the rules given by Sluse; the second, namely, by substitution of $x + dx$ for x, etc., being the more probable. Is Leibniz trying to draw a red herring across the trail, the real trail that leads to Barrow's a and e?

11 July 1677.

Méthode générale pour mener les touchantes des Lignes Courbes sans calcul, et sans réduction des quantités irrationelles et rompues.

[General method for drawing tangents to curves without calculation,and without reducing irrational or fractional quantities.]

Slusius has published his method of finding tangents to curves without calculation, in which the equation is purged of irrational or fractional quantities.



For example, a curve DC being given, in which the equation expresses the relation between BC and AS, which we will call y, and AB or SC, which we will call x; let this be

$$a + bx + cy + dxy + ex^2 + fy^2 + gx^2y + hxy^2 + kx^3 + ly^3 + \text{etc.} = 0.$$

One has only to write

$$\begin{aligned} 0 \;+\; & b\xi + cv + dxv + 2ex\xi + 2fyv + gx^2v + hy^2\xi + 3kx^2\xi + 3ly^2v \\ & \quad dy\xi \qquad\qquad\quad 2gxy\xi \quad 2hxyv \\ & + mx^2y^2 + nx^3y + py^3 + qx^4 + ry^4 \qquad (65) \\ & + 2mx^2yv + nx^3v + py^3\xi + 4qx^3\xi + 4ry^3v \\ & + 2mxy^2\xi + 3nx^2y\xi + 3py^2xv \end{aligned}$$

that is to say, if the equation is changed to a proportion,

$$\frac{\xi}{v} = \frac{c + dx + 2fy + gx^2 + 2hxy + 3ly^2 + 2mx^2y + \text{etc.}}{b + dy + 2ex + 2gxy + hy^2 + 3kx^2 + \text{etc.}};$$

and, supposing that $\frac{\xi}{v}$ expresses the ratio $\frac{\text{TB}}{\text{BC}=x}$ or $\frac{\text{CS}=y}{\text{SV}}$, then TB or SV can be obtained, if BC and SC are supposed to be given. When the given magnitudes, b, c, d, e, etc., with their proper signs, make the value of ξ/v a negative magnitude, the tangent will not be CT which goes toward A, the start of the abscissa AB, but C(T) which goes away from it. That is all that has been

[65] This line represents the "etc." of the original equation, and is set down for the purpose of getting the derived terms; the complete derived equation therefore consists of the two lines above and the two below. Note the omission of the negative sign, when changing from the equation to the proportion.

published up to the present time, easy to understand by any one that is versed in these matters. But when there are irrational or fractional magnitudes, which contain either x or y or both, this method cannot be used, except after a reduction of the given equation to another that is freed from these magnitudes. But at times this increases to a terrible degree the calculation and obliges us to rise to very high dimensions, and leads us to equations for which the process of depression is often very difficult. I have no doubt that the gentlemen[66] I have just named know the remedy that it is necessary to apply, but as it is not as yet in common use, and is I believe known to but a few, also because it gives the finishing touch to the problem that Descartes said was the most difficult to solve of all geometrical problems, because of its general utility, I have thought it a good thing to publish it.

Suppose we have any formula or magnitude or equation such as was given above,

$$a + bx + cy + dxy + ex^2 + fy^2 + \text{etc.};$$

for brevity let us call it ω; that which arises from it when it is treated in the manner given above, namely,

$$b\xi + cv + dxv + dy\xi + \text{ etc.};$$

will be called $d\omega$; and in the same way, if the formula is λ or μ, then the result above will be $d\lambda$ or $d\mu$, and similarly for everything else. Now let the formula or equation or magnitude ω be equal to λ/μ, then I say that $d\omega$ will be equal to $\frac{\mu\overline{d\lambda} - \lambda\overline{d\mu}}{\mu^2}$. This will be sufficient to deal with fractions.

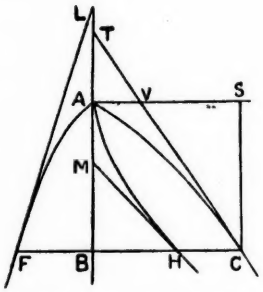
Again, let ω be equal to $\sqrt[z]{\omega}$, then $d\omega = \frac{dw}{z.\sqrt[z]{\omega}^{z-1}}$; and this will be sufficient for the proper treatment of irrationals.

Algorithm of the new analysis for maxima and minima, and for tangents.

Let $AB = x$, and $BC = y$, and let TVC be the tangent to the curve AC; then the ratio $\frac{TB}{BC = y}$ or $\frac{SC = x}{SV}$ will be called $\frac{dx}{dy}$.

[66] Leibniz, at the beginning, first wrote, "Hudde, Sluse, and others"; but later he struck out all but Sluse. (Gerhardt.)

Let there be two or more other curves, AF, AH, and suppose



that $BF = v$ and $BH = w$, and that the straight line FL is the tangent to the curve AF, and MH to the curve AH; also $\frac{LF}{FB} = \frac{dx}{dv}$, and $\frac{MH}{BH} = \frac{dx}{dw}$; then I say that dy, or dvw, will be equal to $vdw + wdv$; and if $v = w = x$, and $y = vw = x^2$, then by substituting x for v and for w, we shall have $dvw = 2xdx$.

(This will also hold good if the angle ABC is either acute or obtuse; also if it is infinitely obtuse, that is to say, if TAC is a straight line.)

[Of this rough draft there is the following revision, and this obviously comes within the same period. (Gerhardt.)]

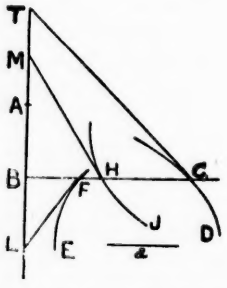
Fermat was the first to find a method which could be made general for finding the straight lines that touch analytical curves. Descartes accomplished it in another way, but the calculation that he prescribes is a little prolix. Hudde has found a remarkable abridgment by multiplying the terms of the progression by those of the arithmetical progression. He has only published it for equations in one unknown; although he has obtained it for those in two unknowns. Then the thanks of the public are due to Sluse; and after that, several have thought that this method was completely worked out. But all these methods that have been published suppose that the equation *has been reduced* and cleared of fractions and irrationals; I mean of those in which the variables occur. I however have found means of obviating these useless reductions, which make the calculation increase to a terrible degree, and oblige us to rise to very high dimensions, in which case we have to look

for a corresponding depression with much trouble; instead of all this, everything is accomplished at the first attack.

This method has more advantage over all the others that have been published, than that of Sluse has over the rest, because it is one thing to give a simple abridgment of the calculation, and quite another thing to get rid of reductions and depressions. With respect to the publication of it, on account of the great extension of the matter which Descartes himself has stated to be the most useful part of Geometry, and of which he has expressed the hope that there is more to follow—in order to explain myself shortly and clearly, I must introduce some *fresh characters*, and give to them a *new Algorithm*, that is to say, altogether special rules, for their addition, subtraction, multiplication, division, powers, roots, and also for equations.

Explanation of the characters.

Suppose that there are several curves, as CD, FE, HJ, connected with one and the same axis AB by ordinates drawn through one and the same point B, to wit, BC, BF, BH. The tangents CT, FL, HM to these curves cut the axis in the points T, L, M; the



point A in the axis is fixed, and the point B changes with the ordinates. Let $AB=x$, $BC=y$, $BF=w$, $BH=v$; also let the ratio of TB to BC be called that of dx to dy, and the ratio of LB to BF that of dx to dw, and the ratio of MB to BH that of dx to dv. Then if, for example, y is equal to vw, we should say dvw instead of dy, and so on for all other cases. Let a be a constant straight line; then, if y is equal to a, that is, if CD is a straight line parallel to AB, dy or da will be equal to 0, or equal to zero. If the magnitude dx/dw comes out negative, then FL, instead of being drawn

toward A, above B, will be drawn in the contrary direction, below B.

Addition and Subtraction. Let $y=v\pm w(\pm)a$, then $\overline{dy}$ will be equal to $\overline{dv}\pm\overline{dw}(\pm)0$.

Multiplication. Let y be equal to avw, then $\overline{dy}$ or $\overline{davw}$ or $a\,\overline{dvw}$ will be equal to $av\,\overline{dw}+aw\,\overline{dv}$.

Division. Let y be equal to $\frac{v}{aw}$, then $\overline{dy}$ or $\overline{d\frac{v}{aw}}$

or $\frac{1}{a}\overline{d\frac{v}{w}}$ will be equal to $\frac{w\,\overline{dv}-v\,\overline{dw}}{aw^2}$.

The rules for *Powers* and *Roots* are really the same thing.

Powers. If $y=w^z$, (where z is supposed to be a certain number), then $\overline{dy}$ will be equal to $z,\ w^{z-1},\ dw$.

Roots or extractions. If $y=\sqrt[z]{w}$, then $\overline{dz}=z\frac{dw}{\frac{z-1}{\sqrt[z]{w}}}$.

Equations expressed in rational integral terms.

$$a+bv+cy+tvy+ev^2+fy^2+gv^2y+hvy^2+kv^3+ly^3$$
$$+mv^2y^2+nv^3y+pvy^3+qv^4+ry^4=0,$$

supposing that a, b, c, t, e, etc. are magnitudes that are known and determined; then we should have

$$0=b\overline{dv}+c\overline{dy}+tv\overline{dy}+2ev\overline{dv}+2fy\overline{dy}+gv^2\overline{dy}+hy^2\overline{dv}$$
$$ty\overline{dv}+2gvy\overline{dy}+2hvy\overline{dy}$$
$$+3ly^2\overline{dy}+2mv^2y\overline{dy}+nv^3\overline{dy}+py^3\overline{dv}+4qv^3\overline{dv}+4ry^3\overline{dy}$$
$$+2mvy^2\overline{dv}+3nv^2y\overline{dv}+3py^2v\overline{dy}$$

This rule can be proved and continued without limit by the preceding rules; for, if

$$a+bv+cy+tvy+ev^2+fy^2+gv^2y+\text{etc.}=0,$$

then $da+dbv+dcy+tdvy+edv^2+fdy^2+gdv^2y+$ etc. will also be equal to 0. Now $da=0$, $dbv=bdv$, $dcy=cdy$, $dvy=vdy+ydv$; also $dv^2=2vdv$, since dv^z is equal to z,v^{z-1},dv, that is to say (by substituting 2 for z) $2vdv$; and $dv^2y=v^2dy+2vydv$, for, supposing that $w=v^2$, then dv^2y will be dwy, and $dwy=ydw+wdy$, and dw or $dv^2=2vdv$; hence in the value of dwy, substituting for w and dw the values found

for them, we shall have $dv^2y = v^2dy + 2vydv$, as obtained above. This can go on without limit. If in the given equation $a + bv + cy$ + etc. $= 0$, the magnitude v were equal to x, that is to say if the line JH were a straight line which when produced passed through the point A, making an angle of 45 degrees with the axis, then the resulting equation, transformed into a proportion, would give the rule for the method of tangents, as published by Sluse; and, in consequence, this is nothing but a particular case or corollary of the general method.

Equations complicated in any manner with fractions and irrationals. These could be treated in the same way without any calculation, by supposing that the denominator of the fraction or the magnitude of which it is necessary to take the root is equal to a magnitude or letter, which is to be treated according to the preceding rules.[67]

Also, when there are magnitudes which have to be multiplied by one another, there is no need to make this multiplication in reality, which saves still more labor. One example will be sufficient.

[No example is given, however; but the following seems to have been added later, according to Gerhardt.]

Lastly this method holds good when the curves are not purely analytical, and even when their nature is not expressed by such ordinates, and in addition it gives a marvelous facility for making geometrical constructions. The true reason for an abridgment so admirable, and one that enables us to avoid reductions of fractions and irrationals, is that one can always make certain, by means of the preceding rules, that the letters dy, dv, dw, and the like, shall not occur in the denominator of the fraction, or under the root-sign.

§ XVI.

The next manuscript appears to be a more detailed revision of the one last considered. It bears no date; but it is safe to say that it belongs to a considerably later period than that of July 1677. For in this are given, by means of the *infinitely small* quantities dx and dy, proofs of the

[67] The complete statement of the method of substitutions.

fundamental rules for the first time; the figure notation is changed from the clumsy C, (C), ((C)) to the neat ${}_1C$, ${}_2C$, ${}_3C$; the notation for proportion is now $a:b::c:d$; and there are several other changes that readers will notice as they go along. The ideas of Leibniz are now approaching crystallization, as is evidenced by the fact that $\int y\,dx$ is clearly stated for the first time to be the sum of *rectangles made from y and dx*. It is rather astonishing, however, in this connection to find $\int \overline{x+y-v} = \int x + \int y - \int v$, which can have no significance according to the above definition; and also to find the whole thing explained by arithmetical series, in which however it is to be observed that dx is not taken to be constant. But for this one might almost place this later than the publication of the method in the *Acta Eruditorum* in 1684; in this essay Leibniz gave a full account of his rules without proofs, and is evidently trying to get away from the idea of the infinitely small, an effort which culminates in the next, and last, manuscript of this set.

If then we guess the date to be about 1680, probably we shall not be very far out.

A remarkable feature of this manuscript is the omission of really necessary figures, without which the text is very hard to follow. Of course this manuscript was written for publication, and the suggestion may be made that the diagrams were drawn separately, just as in books of that time they were printed separately on folding plates; but then, why has he given three diagrams? The only other suggestion that can be made as far as I can see is that he was referring to texts, in which the diagrams were already drawn, by Gregory St. Vincent, Cavalieri, James Gregory (one of whose theorems he quotes), Barrow (who strangely enough also quotes the very same theorem), Wallis, and others. For he mentions many of these authors, but there

is never a word about Barrow. I consider that he was looking up their theorems to show *how much superior his method was to any of theirs.*

It is to be observed that not even in this manuscript is there any mention of logarithms, exponentials, or trigonometrical ratios. We shall see later that Leibniz is reduced to obtaining the integral of $(a^2 + x^2)^{1/2}$ by reference to a figure and its quadrature; that is to say, he is apparently unable to perform the integration analytically. It therefore follows that, if he got a great deal from Barrow, he was unable to understand the Lect. XII, App. I of the *Lectiones Geometricae.*

The final conclusion that I personally have come to, after completing this examination of the manuscripts of Leibniz, as far as they are given by Gerhardt is this:

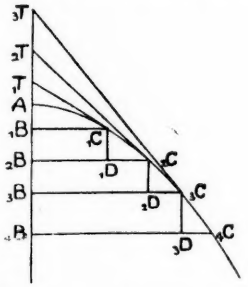
As far as the actual invention of the calculus as he understood the term is concerned, Leibniz received no help from Newton or Barrow; but for the ideas which underlay it, he obtained from Barrow a very great deal more than he acknowledged, and a very great deal less than he would like to have got, or in fact would have got if only he had been more fond of the geometry that he disliked. For, although the Leibnizian calculus was at the time of this essay far superior to that of Barrow on the question of useful application, it was far inferior in the matter of completeness.

(No date.)

Elementa calculi novi pro differentiis et summis, tangentibus et quadraturis, maximis et minimis, dimensionibus linearum, superficierum, solidorum, aliisque communem calculum transcendentibus.

[The elements of the new calculus for differences and sums, tangents and quadratures, maxima and minima, dimensions of lines, surfaces, and solids, and for other things that transcend other means of calculation.]

Let CC be a line, of which the axis is AB, and let BC be ordinates perpendicular to this axis, these being called y, and let AB be the abscissae cut off along the axis, these being called x.



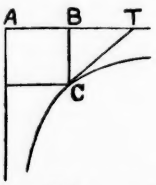
Then CD, the differences of the abscissae, will be called dx; such are ${}_1C\,{}_1D$, ${}_2C\,{}_2D$, ${}_3C\,{}_3D$, etc. Also the straight lines ${}_1D\,{}_2C$, ${}_2D\,{}_3C$, ${}_3D\,{}_4C$, the differences of the ordinates, will be called dy. If now these dx and dy are taken to be infinitely small, or the two points on the curve are understood to be at a distance apart that is less than any given length, i. e., if ${}_1D\,{}_2C$, ${}_2D\,{}_3C$, etc. are considered as the momentaneous increments[68] of the line BC, increasing continuously as it descends along AB, then it is plain that the straight line joining these two points, ${}_2C\,{}_1C$ say, (which is an element of the curve or a side of the infinite-angled polygon that stands for the curve), when produced to meet the axis in ${}_1T$, will be the tangent to the curve, and ${}_1T\,{}_1B$ (the interval between the ordinate and the tangent, taken along the axis) will be to the ordinate ${}_1B\,{}_1C$ as ${}_1C\,{}_1D$ is to ${}_1D\,{}_2C$; or, if ${}_1T\,{}_1B$ or ${}_2T\,{}_2B$, etc. are in general called t, then $t : y :: dx : dy$. Thus to find the differences of series is to find tangents.

For example, it is required to find the tangent to the hyperbola. Here, since $y = \frac{aa}{x}$, supposing that in the diagram, x stands for AB the abscissa along an asymptote, and a for the side of the power, or of the area of the rectangle AB.BC; then

$$dy = -\frac{aa}{xx}dx,$$

[68] Leibniz has evidently seen Newton's work at the time of this composition; also the use of the word "descends" in the next line again suggests Barrow, while the figure is exactly like the top half of the diagram given by Barrow for Lect. XI, 10, which is the theorem of Gregory that is quoted by Leibniz also. For this figure, see the note to that passage.

as will be soon seen when we set forth the method of this calculus; hence $dx : dy$ or $t : y :: -xx : aa :: -x : \frac{aa}{x} :: -x : y$; therefore $t = -y$,



that is, in the hyperbola BT will be equal to AB, but on account of the sign $-x$, BT must be taken not toward A but in the opposite direction.

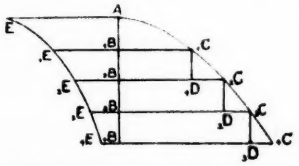
Moreover, differences are the opposite to sums; thus ${}_4B\,{}_4C$ is the sum of all the differences such as ${}_3D\,{}_4C$, ${}_2D\,{}_3C$, etc. as far as A, even if they are infinite in number. This fact I represent thus, $\int dy = y$. Also I represent the area of a figure by the sum of all the rectangles contained by the ordinates and the differences of the abscissae, i. e., by the sum ${}_1B\,{}_1D + {}_2B\,{}_2D + {}_3B\,{}_3D +$ etc. For the narrow triangles ${}_1C\,{}_1D\,{}_2C$, ${}_2C\,{}_2D\,{}_3C$, etc., since they are infinitely small compared with the said rectangles, may be omitted without risk; and thus I represent in my calculus the area of the figure by $\int y\,dx$, or the sum of the rectangles contained by each y and the dx that corresponds to it; here, if the dx's are taken equal to one another, the method of Cavalieri is obtained.

But we, now mounting to greater heights, obtain the area of a figure by finding the figure of its summatrix or quadratrix; and of this indeed the ordinates are to the ordinates of the given figure in the ratio of sums to differences; for instance, let the curve of the figure required to be squared be EE, and let the ordinates to it, EB, which we will call e, be proportional to the differences of the ordinates BC, or to dy; that is let ${}_1B\,{}_1E : {}_2B\,{}_2E :: {}_1D\,{}_2C : {}_2D\,{}_3C$, and so on; or again, let ${}_AB : {}_1B\,{}_1C$, ${}_1C\,{}_1D : {}_1D\,{}_2C$, etc., or $dx : dy$ be in the ratio of a constant or never-varying straight line a to ${}_1B\,{}_1E$ or e; then we have

$$dx : dy :: a : e, \text{ or } e\,dx = a\,dy;$$
$$\therefore \int e\,dx = \int a\,dy.$$

But $e\,dx$ is the same as e multiplied by its corresponding dx, such as the rectangle ${}_3B\,{}_4E$, which is formed from ${}_3B\,{}_3E$ and ${}_3B\,{}_4B$; hence, $\int e\,dx$ is the sum of all such rectangles, ${}_3B\,{}_4E + {}_2B\,{}_1E + {}_3B\,{}_2E$ + etc., and this sum is the figure ${}_AB\,{}_4B\,{}_4EA$, if it is supposed that the

dx's, or the intervals between the ordinates e, or BC, are infinitely small. Again, $a\,dy$ is the rectangle contained by a and dy, such as is contained by ${}_3D\,{}_4C$ and the constant length a, and the sum of



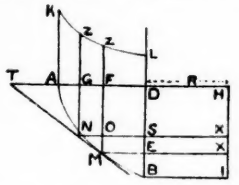
these rectangles, namely $\int a\,dy$, or ${}_3D\,{}_4C.a+{}_2D\,{}_3C.a+{}_1D\,{}_2C.a+$ etc. is the same as ${}_3D\,{}_4C+{}_2D\,{}_3C+{}_1D\,{}_2C+$ etc. into a, that is, the same as ${}_4B\,{}_4C.a$; therefore we have $\int a\,dy = a\int dy = ay$. Therefore $\int e\,dx = ay$, that is, the area $A\,{}_4B\,{}_4EA$ will be equal to the rectangle contained by ${}_4B\,{}_4C$ and the constant line a, and generally ABEA is equal to the rectangle contained by BC and a.[69]

Thus, for quadratures it is only necessary, being given the line EE, to find the summatrix line CC, and this indeed can always be found by calculus, whether such a line is treated in ordinary geometry or whether it is transcendent and cannot be expressed by algebraical calculation; of this matter in another place.

Now the triangle for the line I call the characteristic of the line, because by its most powerful aid there can be found theorems about the line which are seen to be admirable, such as its length, the surface and solid produced by its rotation, and its center of gravity; for ${}_1C\,{}_2C$ is equal to $\sqrt{dx.dx+dy.dy}$. From this we have

[69] Leibniz does not give a diagram, but it is not difficult to construct his figure from the enunciation that he gives for it. The whole of this paragraph should be compared with the following extract from Barrow (Lect. XI, 19), piece by piece.

"Again, let AMB be a curve of which the axis is AD and let BD be perpendicular to AD; also let KZL be another line such that, when any point M is taken in the curve AB, and through it are drawn MT a tangent to the curve AB, and MFZ parallel to DB, cutting KZ in Z and AD in F, and R is a line of given length, TF: FM = R: FZ. Then the space ADLK is equal to the rectangle contained by R and DB.



For, if DH = R and the rectangle BDHI is completed, and MN is taken to be an indefinitely small arc of the curve AB, and MEX, NOS are drawn parallel to AD; then we have NO: MO = TF: FM = R: FZ;

NO.FZ = MO.R and FG.FZ = ES.EX.

Hence, since the sum of such rectangles as FG.FZ differs only in the least degree from the space ADLK, and the rectangles ES.EX form the rectangle DHIB, the theorem is quite obvious.

at once a method for finding the length of a curve by means of some quadrature; e. g., in the case of the parabola, if $y=\frac{xx}{2a}$, then we have $dy=\frac{xdx}{a}$, and hence ${}_1C\,{}_2C=\frac{dx}{a}\sqrt{aa+xx}$; hence, ${}_1C\,{}_2C : dx$ as the ordinate of the hyperbola $\sqrt{aa+xx}$ is to the constant line a; that is, $\frac{1}{a}\int dx\sqrt{aa+xx}$, a straight line equal to the arc of a parabola, depends on the quadrature of the hyperbola, as has already been found by others; and thus we can derive by the calculus all the most beautiful results discovered by Huygens, Wallis, van Huraet, and Neil.[70]

I said above that $t : y :: dx : dy$; hence we have $t\,dy = y\,dx$, and therefore $\int t\,dy = \int y\,dx$. This equation, enunciated geometrically, gives an elegant theorem due to Gregory.[71] namely that, if BAF is a right angle, and AF = BG, and FG is parallel to AB and equal to BT, that is, ${}_1F\,{}_1G = {}_1B\,{}_1T$, then $\int t\,dy$, or the sum of the rectangles contained by t (e. g., ${}_4F\,{}_4G$ or ${}_4B\,{}_4T$) and dy (${}_3F\,{}_4F$ or ${}_3D\,{}_4C$) is equal to the rectangles ${}_4F\,{}_3G + {}_3F\,{}_2G + {}_2F\,{}_1G$ + etc., or the area of the

[70] All the things given are to be found in Barrow, but his name is not even mentioned.

[71] This is the strangest coincidence of all! For, Barrow also quotes this very same theorem of Gregory, and no other theorem; also it occurs in this very same Lect. XI that has been referred to already! *Leibniz does not give a diagram; nor from his enunciation could I complete the figure required, until I had referred to the figure given by Barrow*!!! The two diagrams are given below for comparison, Barrow's figure being the one referred to in the note above. Query, is Leibniz's figure taken from Gregory's original, which I have not been able to see, or is it the Leibnizian variation of Barrow's?

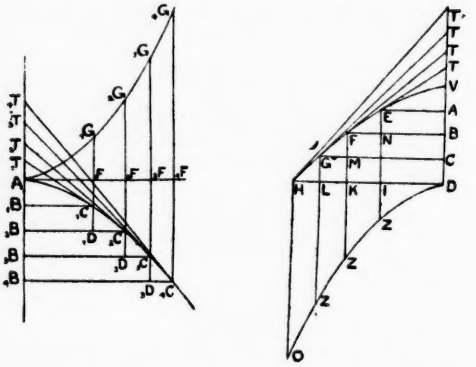


figure A_4F_4GA is equal to $\int y\,dx$, that is, to the figure A_4B_4CA; or generally, the figure AFGA is equal to the figure ABCA.

Again, other things, which are immediately evident on inspection, from a figure, are readily deduced by the calculus; for instance, in the case of the trilinear figure ABCA, the figure ABCA together with its complementary figure AFCA is equal to the rectangle ABCF, for the calculus readily shows that $\int y\,dx + \int x\,dy = xy$.

If it is required to find the volume of the solid formed by rotation round an axis, it is only necessary to find $\int y^2\,dx$; for the solid formed by a rotation round the base, $\int x^2 dy$; for the moment about the vertex, $\int yx\,dx$; and these things serve to find the center of gravity of a figure, and also give the frusta of Gregory St. Vincent, and all that Pascal, Wallis, De Laloubère, and others have found out about these matters.

For, if it is required to find the centers of lines, or the surfaces generated by their rotation, e. g., the surface generated by the rotation of the line AC about AB, it is only necessary to find

$$\int y \sqrt{dx.\,dx + dy.\,dy}$$

or the sum of every PC applied to the axis at the point B that corresponds to it, (thus ${}_2P_2C$ will be applied perpendicular to the axis AB at ${}_2B$), producing in this way a figure of which the above represents the area. Thus the whole thing will immediately reduce to the quadrature of some plane figure, if, instead of y and dy, their values, obtained from the nature of the ordinates and the tangents to the curve, are substituted. Thus, in the case of the parabola, if y is equal to $\sqrt{2ax}$, then $dy = \frac{a\,dx}{y}$ (as will be seen directly); hence we get

$$\int y\sqrt{dxdx + \frac{aa}{yy}\,dxdx} \text{ or } \int dx\sqrt{yy + aa} \text{ or } \int dx\sqrt{2ax + aa},$$

which depends on the quadrature of the parabola (for every $\sqrt{2ax + aa}$ or PC can be applied to a parabola, if it is supposed that AC is the parabola, and AB its axis, provided in that case the figure is changed and the curve turns its concavity toward the axis);[72] and this may be obtained by ordinary geometry, and there-

[72] The Latin here is rather ambiguous; query, a misprint. But I think I have correctly rendered the argument. It is to be noted that the parabola was at this period always thought of in the form we should now denote by the equation $y = x^2$, and the figure referred to by Leibniz is that which Wallis calls the complement of the semiparabola.

UMI

fore also a circle will be found equal to the surface of the parabolic conoid; but this is not the place to deduce it at full length.

Now these, which may seem to be great matters, are only the very simplest results to be obtained by this calculus; for many much more important consequences follow from it, nor does there occur any simple problem in geometry, either pure or applied to mechanics, that can altogether evade its power. Now we will expound the elements of the calculus itself.

The fundamental principle of the calculus.

Differences and sums are the inverses of one another, that is to say, the sum of the differences of a series is a term of the series, and the difference of the sums of a series is a term of the series; and I enunciate the former thus, $\int dx = x$, and the latter thus, $d\int x = x$.

Thus, let the differences of a series, the series itself, and the sums of the series, be, let us say,

Diffs.		1	2	3	4	5	.....	dx
Series	0	1	3	6	10	15	....	x
Sums		0	1	4	10	20	25 ..	$\int x$

Then the terms of the series are the sums of the differences, or $x = \int dx$; thus, $3 = 1+2$, $6 = 1+2+3$, etc.; on the other hand, the differences of the sums of the series are terms of the series, or $d\int x = x$; thus, 3 is the difference between 1 and 4, 6 between 4 and 10.

Also $da = 0$, if it is given that a is a constant quantity, since $a - a = 0$.

Addition and Subtraction.

The difference or sum of a series, of which the general term is made up of the general terms of other series by addition or subtraction, is made up in exactly the same manner from the differences or sums of these series; or

$$x + y - v = \int dx + dy - dv, \quad \int x + y - v = \int x + \int y - \int v.$$

This is evident at sight, if you take any three series, set out their sums and their differences, and take them together correspondingly as above.

Simple Multiplication.

Here $dxy = x\,dx + y\,dy$, or $xy = \int x\,dx + \int y\,dy$.

This is what we said above about figures taken together with their complements being equal to the circumscribed rectangle. It is demonstrated by the calculus as follows:

dxy is the same thing as the difference between two successive xy's; let one of these be xy, and the other $x+dx$ into $y+dy$; then we have

$$dxy = \overline{x+dx}\,.\,\overline{y+dy} - xy = x\,dy + y\,dx + dx\,dy;$$

the omission of the quantity $dx\,dy$, which is infinitely small in comparison with the rest, for it is supposed that dx and dy are infinitely small (because the lines are understood to be continuously increasing or decreasing by very small increments throughout the series of terms), will leave $x\,dy + y\,dx$; the signs vary according as y and x increase together, or one increases as the other decreases; this point must be noted.

Simple Division.

Here we have $d\frac{y}{x} = \frac{x\,dy - y\,dx}{xx}$.

For, $d\frac{y}{x} = \frac{y+dy}{x+dx} - \frac{y}{x} = \frac{x\,dy - y\,dx}{xx + x\,dx}$, which becomes (if we write xx for $xx + x\,dx$, since $x\,dx$ can be omitted as being infinitely small in comparison with xx) equal to $\frac{x\,dy - y\,dx}{xx}$; also, if $y = aa$, then $dy = 0$, and the result becomes $-\frac{aadx}{xx}$, which is the value we used a little while before in the case of the tangent to the hyperbola.

From this any one can deduce by the calculus the rules for *Compound Multiplication and Division*; thus,

$$dxvy = xy\,dv + xv\,dy + yv\,dx,$$

$$d\frac{y}{vz} = \frac{xv\,dy - yv\,dz - yz\,dv}{vv.zz};$$

as can be proved from what has gone before; for we have

$$d\frac{y}{x} = \frac{x\,dy - y\,dx}{xx};$$

hence, putting zv for x, and $z\,dv + v\,dz$ for dx or dzv in the above, we obtain what was stated.

UMI

Powers follow: $dx^2 = 2x\,dx$, $dx^3 = 3x^2\,dx$, and so on. For, putting $y = x$, and $v = x$, we can write dx^2 for dxy, and this is (from above) equal to $x\,dy + y\,dx$, or (if $x = y$, and consequently $dx = dy$) equal to $2x\,dx$. Similarly, for dx^3 we write $dxyv$, that is (from above) $xy\,dv + xv\,dy + yv\,dx$, or (putting x for y and v and dx for dy and dv) equal to $3x^2dx$. Q. E. D. By the same method, in general, $dx^e = e.x^{e-1}\,dx$, as can easily be proved from what has been said.

Hence also, $$d\frac{1}{x^h} = -\frac{h\,dx}{x^{h+1}}.$$

For, if $\frac{1}{x^h} = x^e$, then $e = -h$, and $x^{e-1} = \frac{1}{x^{h+1}}$, as is well known to any one who understands the nature of the exponents in a geometrical progression. The same thing will do for *fractions*. The procedure is the same for irrationals or *Roots*. $d\sqrt[r]{x^h} = dx^{h:r}$, (where by $h:r$ I mean h/r, or h divided by r), or dx^e (taking e equal to h/r), or $e.x^{e-1}\,dx$, by what has been said above, or (by substituting once more $h:r$ for e, and $\overline{h-r}:r$ for $e-1$) $\frac{h}{r}.\,x^{\overline{h-r}:r}.dx$; and thus finally we get the value of $d\sqrt[r]{x^h}$.

Moreover, conversely, we have

$$\int x^e dx = \frac{x^{e+1}}{e+1},\quad \int \frac{1}{x^e}dx = -\frac{1}{e-1.x^{e-1}},\quad \int \sqrt[r]{x^h}.dx = \frac{r}{r+h}\sqrt[r]{x^{\overline{h+r}:r}}.$$

These are the elementary principles of the differential and summatory calculus, by means of which highly complicated formulas can be dealt with, not only for a fraction or an irrational quantity, or anything else; but also an indefinite quantity, such as x or y, or any other thing expressing generally the terms of any series, may enter into it.

§ XVII.

The next manuscript bears no date; but this can be easily assigned to a certain extent, from internal evidence. It is for one thing later than the publication in the *Acta Eruditorum* of Leibniz's first communication to the world of his calculus in 1684. The manuscript is an answer, or rather the first rough draft probably of such an answer, to the animadversions of Bernhard Nieuwentijt against the idea of the infinitesimal calculus. The latter stated that (i) Leibniz could explain no more than Barrow or

Newton how the infinitely small differences differed from absolute zero; (ii) it was not clear how the differentials of higher order were obtained from those of the first order; (iii) the differential method cannot be applied to exponential functions. Leibniz answers the first point skilfully, fails over the second through erroneous work, which I think he afterward perceived; for he has a note that the whole thing is to be carefully revised before publication. It almost seems that he was not quite confident in his own powers of completely answering these objections, for he also notes that the rudeness of language in which the answer is commenced must be mollified.

On the third point he is silent; in the later written *Historia,* we have seen he is able to get, not over, but round the difficulty of the exponential function; but the silence here would seem to say that Leibniz could not manage exponentials as yet.

The success of the answer to the first point is due to the underlying principle that the ratio $dy : dx$ ultimately becomes a *rate*; when this idea is muddled by an admixture of the infinitesimal idea in the last paragraph the result is almost disastrous. Leibniz, however, looked on his calculus as a tried tool more than anything else.

When my infinitesimal calculus, which includes the calculus of differences and sums, had appeared and spread, certain over-precise veterans began to make trouble; just as once long ago the Sceptics opposed the Dogmatics, as is seen from the work of Empicurus against the mathematicians (i. e., the dogmatics), and such as Francisco Sanchez, the author of the book *Quod nihil scitur,* brought against Clavius; and his opponents to Cavalieri, and Thomas Hobbes to all geometers, and just lately such objections as are made against the quadrature of the parabola by Archimedes by that renowned man, Dethlevus Cluver. When then our method of infinitesimals, which had become known by the name of the calculus of differences, began to be spread abroad by several examples of its use, both of my own and also of the famous brothers Bernoulli, and more espe-

UMI

cially by the elegant writings of that illustrious Frenchman, the Marquis d'Hopital, just lately a certain erudite mathematician, writing under an assumed name in the scientific *Journal de Trevoux*, appeared to find fault with this method. But to mention one of them by name, even before this there arose against me in Holland Bernard Nieuwentiit, one indeed really well equipped both in learning and ability, but one who wished rather to become known by revising our methods to some extent than by advancing them. Since I introduced not only the first differences, but also the second, third and other higher differences, inassignable or incomparable with these first differences, he wished to appear satisfied with the first only; not considering that the same difficulties existed in the first as in the others that followed, nor that wherever they might be overcome in the first, they also ceased to appear in the rest. Not to mention how a very learned young man, Hermann of Basel, showed that the second and higher differences were avoided by the former in name only, and not in reality; moreover, in demonstrating theorems by the legitimate use of the first differences, by adhering to which he might have accomplished some useful work on his own account, he fails to do so, being driven to fall back on assumptions that are admitted by no one; such as that something different is obtained by multiplying 2 by m and by multiplying m by 2; that the latter was impossible in any case in which the former was possible; also that the square or cube of a quantity is not a quantity or Zero.

In it, however, there is something that is worthy of all praise, in that he desires that the differential calculus should be strengthened with demonstrations, so that it may satisfy the rigorists; and this work he would have procured from me already, and more willingly, if, from the fault-finding everywhere interspersed, the wish had not appeared foreign to the manner of those who desire the truth rather than fame and a name.

It has been proposed to me several times to confirm the essentials of our calculus by demonstrations, and here I have indicated below its fundamental principles, with the intent that any one who has the leisure may complete the work. Yet I have not seen up to the present any one who would do it. For what the learned Hermann has begun in his writings, published in my defence against Nieuwentiit, is not yet complete.

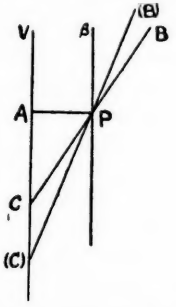
For I have, beside the mathematical infinitesimal calculus, a method also for use in Physics, of which an example was given in

the *Nouvelles de la République des Lettres*; and both of these I include under the Law of Continuity; and adhering to this, I have shown that the rules of the renowned philosophers Descartes and Malebranche were sufficient in themselves to attack all problems on Motion.

I take for granted the following postulate:

In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included.

For example, if A and B are any two quantities, of which the former is the greater and the latter is the less, and while B remains the same, it is supposed that A is continually diminished, until A becomes equal to B; then it will be permissible to include under a general reasoning the prior cases in which A was greater than B, and also the ultimate case in which the difference vanishes and A is equal to B. Similarly, if two bodies are in motion at the same time, and it is assumed that while the motion of B remains the same, the velocity of A is continually diminished until it vanishes altogether, or the speed of A becomes zero; it will be permissible to include this case with the case of the motion of B under one general reasoning. We do the same thing in geometry, when two



straight lines are taken, produced in any manner, one VA being given in position or remaining in the same site, the other BP passing through a given point P, and varying in position while the point P remains fixed; at first indeed converging toward the line VA and meeting it in the point C; then, as the angle of inclination VCA is continually diminished, meeting VA in some more remote point (C), until at length from BP, through the position (B)P, it comes

to βP, in which the straight line no longer converges toward VA, but is parallel to it, and C is an impossible or imaginary point. With this supposition it is permissible to include under some one general reasoning not only all the intermediate cases such as (B)P but also the ultimate case βP.

Hence also it comes to pass that we include as one case ellipses and the parabola, just as if A is considered to be one focus of an ellipse (of which V is the given vertex), and this focus remains fixed, while the other focus is variable as we pass from ellipse to ellipse, until at length (in the case when the line BP, by its intersection with the line VA, gives the variable focus) the focus C becomes evanescent[73] or impossible, in which case the ellipse passes into a parabola. Hence it is permissible with our postulate that a parabola should be considered with ellipses under a common reasoning. Just as it is common practice to make use of this method in geometrical constructions, when they include under one general construction many different cases, noting that in a certain case the converging straight line passes into a parallel straight line, the angle between it and another straight line vanishing.

Moreover, from this postulate arise certain expressions which are generally used for the sake of convenience, but seem to contain an absurdity, although it is one that causes no hindrance, when its proper meaning is substituted. For instance, we speak of an imaginary point of intersection as if it were a real point, in the same manner as in algebra imaginary roots are considered as accepted numbers. Hence, preserving the analogy, we say that, when the straight line BP ultimately becomes parallel to the straight line VA, even then it converges toward it or makes an angle with it, only that the angle is then infinitely small; similarly, when a body ultimately comes to rest, it is still said to have a velocity, but one that is infinitely small; and, when one straight line is equal to another, it is said to be unequal to it, but that the difference is infinitely small; and that a parabola is the ultimate form of an ellipse, in which the second focus is at an infinite distance from the given focus nearest to the given vertex, or in which the ratio of PA to AC, or the angle BCA, is infinitely small.

Of course it is really true that things which are absolutely equal have a difference which is absolutely nothing; and that straight lines which are parallel never meet, since the distance

[73] The term is here used with the idea of "vanishing into the far distance."

between them is everywhere the same exactly; that a parabola is not an ellipse at all, and so on. Yet, a state of transition may be imagined, or one of evanescence, in which indeed there has not yet arisen exact equality or rest or parallelism, but in which it is passing into such a state, that the difference is less than any assignable quantity; also that in this state there will still remain some difference, some velocity, some angle, but in each case one that is infinitely small; and the distance of the point of intersection, or the variable focus, from the fixed focus will be infinitely great, and the parabola may be included under the heading of an ellipse (and also in the some manner and by the same reasoning under the heading of a hyperbola), seeing that those things that are found to be true about a parabola of this kind are in no way different, for any construction, from those which can be stated by treating the parabola rigorously.

Truly it is very likely that Archimedes, and one who seems so have surpassed him, Conon, found out their wonderfully elegant theorems by the help of such ideas; these theorems they completed with *reductio ad absurdum* proofs, by which they at the same time provided rigorous demonstrations and also concealed their methods. Descartes very appropriately remarked in one of his writings that Archimedes used as it were a kind of metaphysical reasoning (Caramuel would call it metageometry), the method being scarcely used by any of the ancients (except those who dealt with quadratrices); in our time Cavalieri has revived the method of Archimedes, and afforded an opportunity for others to advance still further. Indeed Descartes himself did so, since at one time he imagined a circle to be a regular polygon with an infinite number of sides, and used the same idea in treating the cycloid; and Huygens too, in his work on the pendulum, since he was accustomed to confirm his theorems by rigorous demonstrations; yet at other times, in order to avoid too great prolixity, he made use of infinitesimals; as also quite lately did the renowned La Hire.

For the present, whether such a state of instantaneous transition from inequality to equality, from motion to rest, from convergence to parallelism, or anything of the sort, can be sustained in a rigorous or metaphysical sense, or whether infinite extensions successively greater and greater, or infinitely small ones successively less and less, are legitimate considerations, is a matter that I own to be possibly open to question; but for him who would discuss these matters, it is not necessary to fall back upon metaphysical

UMI

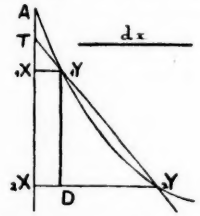
controversies, such as the composition of the continuum, or to make geometrical matters depend thereon. Of course, there is no doubt that a line may be considered to be unlimited in any manner, and that, if it is unlimited on one side only, there can be added to it something that is limited on both sides. But whether a straight line of this kind is to be considered as one whole that can be referred to computation, or whether it can be allocated among quantities which may be used in reckoning, is quite another question that need not be discussed at this point.

It will be sufficient if, when we speak of infinitely great (or more strictly unlimited), or of infinitely small quantities (i. e., the very least of those within our knowledge), it is understood that we mean quantities that are indefinitely great or indefinitely small, i. e., as great as you please, or as small as you please, so that the error that any one may assign may be less than a certain assigned quantity. Also, since in general it will appear that, when any small error is assigned, it can be shown that it should be less, it follows that the error is absolutely nothing; an almost exactly similar kind of argument is used in different places by Euclid, Theodosius and others; and this seemed to them to be a wonderful thing, although it could not be denied that it was perfectly true that, from the very thing that was assumed as an error, it could be inferred that the error was non-existent. Thus, by infinitely great and infinitely small, we understand something indefinitely great, or something indefinitely small, so that each conducts itself as a sort of class, and not merely as the last thing of a class. If any one wishes to understand these as the ultimate things, or as truly infinite, it can be done, and that too without falling back upon a controversy about the reality of extensions, or of infinite continuums in general, or of the infinitely small, ay, even though he think that such things are utterly impossible; it will be sufficient simply to make use of them as a tool that has advantages for the purpose of the calculation, just as the algebraists retain imaginary roots with great profit. For they contain a handy means of reckoning, as can manifestly be verified in every case in a rigorous manner by the method already stated.

But it seems right to show this a little more clearly, in order that it may be confirmed that the algorithm, as it is called, of our differential calculus, set forth by me in the year 1684, is quite reasonable. First of all, the sense in which the phrase "dy is the

element of y," is to be taken will best be understood by considering a line AY referred to a straight line AX as axis.

Let the curve AY be a parabola, and let the tangent at the vertex A be taken as the axis. If AX is called x, and AY, y, and the latus-rectum is a, the equation to the parabola will be $xx = ay$, and this holds good at every point. Now, let $A_1X = x$, and $_1X_1Y = y$



and from the point $_1Y$ let fall a perpendicular $_1YD$ to some greater ordinate $_2X_2Y$ that follows, and let $_1X_2X$, the difference between A_1X and A_2X, be called dx; and similarly, let D_2Y, the difference between $_1X_1Y$ and $_2X_2Y$, be called dy.

Then, since $y = xx : a$, by the same law, we have

$$y + dy = xx + 2x\,dx + dx\,dx, : a ;$$

and taking away the y from the one side and the $xx : a$ from the other, we have left

$$dy : dx = 2x + dx : a ;$$

and this is a general rule, expressing the ratio of the difference of the ordinates to the difference of the abscissae, or, if the chord $_1Y_2Y$ is produced until it meets the axis in T, then the ratio of the ordinate $_1X_1Y$ to T_1X, the part of the axis intercepted between the point of intersection and the ordinate, will be as $2x + dx$ to a. Now, since by our postulate it is permissible to include under the one general reasoning the case also in which the ordinate $_2X_2Y$ is moved up nearer and nearer to the fixed ordinate $_1X_1Y$ until it ultimately coincides with it, it is evident that in this case dx becomes equal to zero and should be neglected, and thus it is clear that, since in this case T_1Y is the tangent, $_1X_1Y$ is to T_1X as $2x$ is to a.

Hence, it may be seen that there is no need in the whole of our differential calculus to say that those things are equal which have a difference that is infinitely small, but that those things can be taken as equal that have not any difference at all, provided that the calculation is supposed to be general, including both the cases in which there is a difference and in which the difference is zero;

and provided that the difference is not assumed to be zero until the calculation is purged as far as is possible by legitimate omissions, and reduced to ratios of non-evanescent quantities, and we finally come to the point where we apply our result to the ultimate case.

Similarly, if $x^3 = aay$, then we have

$$x^3 + 3xx\,dx + 3x\,dx\,dx + dx\,dx\,dx = aay + aa\,dy,$$

or cancelling from each side,

$$3xx\,dx + 3x\,dx\,dx + dx\,dx\,dx = aa\,dy,$$

or $$3xx + 3x\,dx + dx\,dx, : aa = dy : dx = {}_1X\,{}_1Y : T\,{}_1X;$$

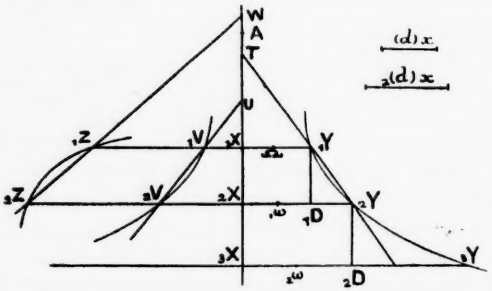
hence, when the difference vanishes, we have

$$3xx : aa = {}_1X\,{}_1Y : T\,{}_1X.$$

But if it is desired to retain dy and dx in the calculation, so that they may represent non-evanescent quantities even in the ultimate case, let any assignable straight line be taken as (dx), and let the straight line which bears to (dx) the ratio of y or ${}_1X\,{}_1Y$ to ${}_1XT$ be called (dy); in this way dy and dx will always be assignables bearing to one another the ratio of D_2Y to D_1Y, which latter vanish in the ultimate case.

[Leibniz here gives a correction for a passage in the *Acta Eruditorum*, which is unintelligible without the context.]

On these suppositions, all the rules of our algorithm, as set out in the *Acta Eruditorum* for October 1684, can be proved without much trouble.



Let the curves YY, VV, ZZ be referred to the same axis AXX; and to the abscissae $A\,{}_1X$ $(=x)$ and $A\,{}_2X$ $(=x+dx)$ let there correspond the ordinates ${}_1X\,{}_1Y$ $(=y)$ and ${}_2X\,{}_2Y$ $(=y+dy)$, and also the ordinates ${}_1X\,{}_1V$ $(=v)$ and ${}_2X\,{}_2V$ $(=v+dv)$, and the ordinates

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${}_1X\,{}_1Z\ (=z)$ and ${}_2X\,{}_2Z\ (=z+dz)$. Let the chords ${}_1Y\,{}_2Y$, ${}_1V\,{}_2V$, ${}_1Z\,{}_2Z$, when produced meet the axis AXX in T, U, W. Take any straight line you will as $(d)x$, and, while the point ${}_1X$ remains fixed and the point ${}_2X$ approaches ${}_1X$ in any manner, let this remain constant, and let $(d)y$ be another line which bears to $(d)x$ the ratio of y to ${}_1XT$, or of dy to dx; and similarly, let $(d)v$ be to $(d)x$ as v to ${}_1XU$ or dv to dx; also let $(d)z$ be to $(d)x$ as z to ${}_1XW$ or dz to dx; then $(d)x$, $(d)y$, $(d)z$, $(d)v$ will always be ordinary or assignable straight lines.

Nor for *Addition and Subtraction* we have the following:

If $y-z=v$, then $(d)y-(d)z=(d)v$.

This I prove thus: $y+dy-z-dz=v+dv$, (if we suppose that as y increases, z and v also increase; otherwise for decreasing quantities, for z say, $-dz$ should be taken instead of dz, as I mentioned once before); hence, rejecting the equals, namely $y-z$ from one side, and v from the other, we have $dy-dz=dv$, and therefore also $dy-dz:dx=dv:dx$. But $dy:dx$, $dz:dx$, $dv:dx$ are respectively equal to $(d)y:(d)x$, $(d)z:(d)x$, and $(d)v:(d)x$. Similarly, $(d)z:(d)y$ and $(d)v:(d)y$ are respectively equal to $dz:dy$ and $dv:dy$. Hence, $(d)y-(d)z,:(d)x=(d)v:(d)x$; and thus $(d)y-(d)z$ is equal to $(d)v$, which was to be proved; or we may write the result as $(d)v:(d)y=1-(d)z:(d)y$.

This rule for addition and subtraction also comes out by the use of our postulate of a common calculation, when ${}_1X$ coincides with ${}_2X$, and ${}_1YT$, ${}_1YU$, ${}_1YW$ are the tangents to the curves YY, VV, ZZ. Moreover, although we may be content with the assignable quantities $(d)y$, $(d)v$, $(d)z$, $(d)x$, etc., since in this way we may perceive the whole fruit of our calculus, namely a construction by means of assignable quantities, yet it is plain from what I have said that, at least in our minds, the unassignables dx and dy may be substituted for them by a method of supposition even in the case when they are evanescent; for the ratio $dy:dx$ can always be reduced to the ratio $(d)y:(d)x$, a ratio between quantities that are assignable or undoubtedly real. Thus we have in the case of tangents $dv:dy=1-dz:dx$, or $dv=dy-dz$.

Multiplication. Let $ay=xv$, then $a(d)y=x(d)v+v(d)x$.

Proof. $ay+a\,dy=x+dx$, $v+dv=xv+x\,dv+v\,dx+dx\,dv$;

and, rejecting the equals ay and xy from the two sides,

$$a\,dy = x dv + v\,dx + dx\,dv,$$

or

$$\frac{a\,dy}{dx} = \frac{x\,dv}{dx} + v + dv;$$

and transferring the matter, as we may, to straight lines that never become evanescent, we have

$$\frac{a(d)y}{(d)x} + \frac{x(d)y}{(d)x} + v + dv;$$

so that, since it alone can become evanescent, dv is superfluous, and in the case of the vanishing differences, as in that case $dv=0$, we have

$$a(d)y = x(d)v + v(d)x, \text{ as was stated,}$$

or $$(d)y : (d)x = x + v, : a.$$

Also, since $(d)y : (d)x$ always $= dy : dx$, it will be allowable to suppose this is true in the case when dy, dx become evanescent, and to say that $dy : dx = x + v : a$, or $a\,dy = x\,dv + v\,dx$.

Division. Let $z : a = v : x$, then $(d)z : a = v(d)x - x(d)y, : xx$.

Proof $$z + dz : a = v + dv, : x + dx;$$

or clearing of fractions, $xz + xdz + zdx + dzdx = av + adv$; taking away the equals xz and av from the two sides, and dividing what is left by dx, we have

$$a\,dv - x\,dz, : dx = z + dz,$$

or $$a(d)v - x(d)z, : dx = z + dz;$$

and thus, only dz, which can become evanescent, is superfluous. Also, in the case of vanishing differences, when ${}_1X$ coincides with ${}_2X$, since in that case $dz=0$, we have

$$a(d)v - x(d)z, : (d)x = z = av : x;$$

whence, (as was stated) $(d)z = ax(d)v - av(d)x, : xx$,

or $$(d)z : (d)x = (a : x)(d)v : (d)x - av : xx.$$

Also, since $(d)z : (d)x$ is always equal to $dz : dx$, on all other occasions, it is allowable to suppose this to be so also when dz, dv, dx are evanescent, and to put

$$dz : dx = ax\,dv - av\,dx, : xx$$

For *Powers*, let the equation be $a^{n-r}x^r = y^n$, then

$$\frac{(d)y}{(d)x} = \frac{e.x^{r-1}}{n.y^{\frac{n-1}{\cdot}}};$$

and this I will prove in a manner a little more detailed than those above, thus:

$$a^{n-e}, \frac{1}{1}x^{e} + \frac{e}{1}x^{e-1}\,dx + \frac{e,e-1}{1,2}x^{e-2}\,dxdx + \frac{e,e-1,e-2}{1,2,3}x^{e-3}\,dxdxdx$$

(and so on until the factor $e-e$ or 0 is reached)

$$= \frac{1}{1}y^{n} + \frac{n}{1}y^{n-1}\,dy + \frac{n,n-1}{1,2}y^{n-2}\,dydy + \frac{n,n-1,n-2}{1,2,3}y^{n-3}\,dydydy$$

(and so on until the factor $n-n$ or 0 is reached);

take away from the one side $a^{n-e}x^{e}$, and from the other side y^{n}, these being equal to one another, and divide what is left by dx, and lastly, instead of the ratio $dy:dx$, between the two quantities that continually diminish, substitute the ratio that is equal to it, $(d)y:(d)x$, a ratio between two quantities, of which one, $(d)x$, always remains the same during the time that the differences are diminishing, or while ${}_2\mathrm{X}$ is approaching the fixed point ${}_1\mathrm{X}$ and we have

$$\frac{e}{1}x^{e-1} + \frac{e,e-1}{1,2}x^{e-2}\,dx + \frac{e,e-1,e-2}{1,2,3}x^{e-3}\,dxdx + \text{etc.}$$

$$= \frac{n}{1}y^{n-1}\frac{(d)y}{(d)x} + \frac{n,n-1}{1,2}y^{n-2}\frac{(d)y}{(d)x}\,dy + \frac{n,n-1,n-2}{1,2,3}y^{n-3}\frac{(d)y}{(d)x}\,dydy + \text{etc.}$$

Now, since by the postulate there is included in this general rule the case also in which the differences become equal to zero, that is when the points ${}_2\mathrm{X}$, ${}_2\mathrm{Y}$ coincide with the points ${}_1\mathrm{X}$, ${}_1\mathrm{Y}$ respectively; therefore, in that case, putting dx and dy equal to 0, we have

$$\frac{e}{1}x^{e-1} = \frac{n}{1}y^{n-1}\frac{(d)y}{(d)x},$$

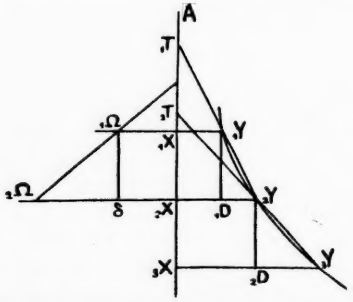
the remaining terms vanishing, or $(d)y : (d)x = e.x^{e-1} : n\,y^{n-1}$. Moreover, as we have explained, the ratio $(d)y:(d)x$ is the same as the ratio of y, or the ordinate ${}_1\mathrm{X}\,{}_1\mathrm{Y}$, to the subtangent ${}_1\mathrm{XT}$, where it is supposed that $\mathrm{T}_1\mathrm{Y}$ touches the curve in ${}_1\mathrm{Y}$.

This proof holds good whether the powers are integral powers or roots of which the exponents are fractions. Though we may also get rid of fractional exponents by raising each side of the equation to some power, so that e and n will then signify nothing else but powers with rational exponents, and there will be no need of a series proceeding to infinity. Moreover, at any rate, it will be permissible, by means of the explanation given above, to return to the unassignable quantities dy and dx, by making in the case of evanescent differences, as in all other cases, the supposition that the ratio of the evanescent quantities dy and dx is equal to the ratio

of $(d)y$ and $(d)x$, because this supposition can always be reduced to an undoubtable truth.

Thus far the algorithm has been demonstrated for differences of the first order: now I will proceed to show that the same method will hold good for the differences of the differences. For this purpose, take three ordinates, ${}_1X\,{}_1Y$, ${}_2X\,{}_2Y$, ${}_3X\,{}_3Y$, of which ${}_1X\,{}_1Y$ remains constant, but ${}_2X\,{}_2Y$ and ${}_3X\,{}_3Y$ continually approach ${}_1X\,{}_1Y$ until finally they both coincide with it simultaneously; which will happen if the speed with which ${}_3X$ approaches ${}_1X$ is to the speed with which ${}_2X$ approaches ${}_1X$ is in the ratio of ${}_1X\,{}_3X$ to ${}_1X\,{}_2X$. Also let two straight lines be assigned, $(d)x$ always constant for any position of ${}_2X$, and ${}_2(d)x$ for any position of ${}_3X$; also let $(d)y$ always be to $(d)x$ as $D\,{}_2Y$ is to ${}_1X\,{}_2X$, or as y (i. e., ${}_1X\,{}_1Y$) is to ${}_1XT$; thus, while $(d)x$ remains always the same, $(d)y$ will be altered as ${}_2X$ approaches ${}_1X$; similarly, let ${}_2(d)y$ be to ${}_2(d)x$ as ${}_2D\,{}_3Y$ to ${}_2X\,{}_3X$ or as $y+dy$ (i. e., ${}_2X\,{}_2Y$) to ${}_2X\,{}_2T$; thus while ${}_2(d)x$ remains constant, ${}_2(d)y$ will be altered as ${}_3X$ approaches ${}_1X$.

Also let $(d)y$ be always taken in the varying line ${}_2X\,{}_2Y$, and let ${}_2X\,{}_1\omega$ be equal to $(d)y$, and similarly take ${}_2(d)y$ in the line ${}_3X\,{}_3Y$, and let ${}_3X\,{}_2\omega$ be equal to ${}_2(d)y$. Thus, while ${}_2X$ and ${}_3X$ continually approach to the straight line ${}_1X\,{}_1Y$, ${}_2X\,{}_1\omega$ and ${}_3X\,{}_2\omega$ continually approach it also, and finally coincide with it at the same time as



${}_2X$ and ${}_3X$. Further, let the point in the ordinate ${}_1X\,{}_1Y$, which ${}_1\omega$ continually approaches and with which it at last coincides, be marked, and let it be Ω; then ${}_1X\Omega$ is the ultimate $(d)y$, which bears to $(d)x$ the ratio of the ordinate ${}_1X\,{}_1Y$ to the subtangent ${}_1XT$, where it is supposed that $T\,{}_1X$ touches the curve in ${}_1Y$, because then indeed ${}_1Y$ and ${}_2Y$ coincide. Now, since all this can be done,

no matter where ${}_1Y$ may be taken on the curve, it is evident that a curve $\Omega\Omega$ will be produced in this way, which is the differentrix of the curve YY; just as, conversely, the curve YY is the summatrix curve of $\Omega\Omega$, as can be readily demonstrated.

By this method, the calculus may be demonstrated also for the differences of the differences.

Let ${}_1X\,{}_1Y$, ${}_2X\,{}_2Y$, ${}_3X\,{}_3Y$ be three ordinates, of which the values are y, $y+dy$, $y+dy+ddy$, and let ${}_1X\,{}_2X\,(dx)$ and ${}_2X\,{}_3X\,(dx+ddx)$ be any distances, and ${}_2D\,{}_3Y\,(dy)$ and ${}_2D\,{}_3Y\,(dy+ddy)$ the differences. Now the difference between $(d)y$ and ${}_2(d)y$, or between ${}_1X\Omega$ and ${}_2X\,{}_2\Omega$ is $\delta\,{}_2\Omega$, and that between ${}_1X\,{}_2X$ and ${}_2X\,{}_3X$ is ddx; also let

$$(d)dx : (d)x = dx : {}_2(d)x, \quad {}^{74} \text{ and similarly let}$$

$$(d)dy : (d)y = {}_2\Omega\delta : {}_1X\,{}_2X \text{ or } {}_1X\Omega : {}_1XT.$$

Now, for the sake of example, let us take $ay = xv$. Then we have $a\,dy = x\,dv + v\,dx + dx\,dv$, as has been shown above; and similarly,

$$\begin{aligned} a\,dy + a\,ddy &= (x+dx)(dv+ddv) + (v+dv)(dx+ddx) \quad {}^{75} \\ &\quad + (dx+ddx)(dv+ddv) \\ &= x\,dv + x\,ddv + dx\,dv + dx\,ddv + v\,dx + v\,ddx \\ &\quad + dv\,dx + dv\,ddx + dx\,dv + dx\,ddv \\ &\quad + ddx\,dv + ddx\,ddv. \end{aligned}$$

Taking away $a\,dy$ from one side, and $x\,dx + v\,dx + dx\,dv$ from the other, there will be left in any case

$$\frac{ddy}{ddx} = \frac{x}{a}\frac{ddy}{ddx} + \frac{v}{a} + \frac{2}{a}\frac{dx\,dv}{ddx} + \frac{2\,dv}{a} + \frac{2\,dx}{a}\frac{ddx}{ddx} + \frac{ddv}{a}.$$

In this it is evident that the ratio between ddy and ddx can be expressed by the ratio of the straight line $(d)dy$ to $(d)x$, the straight line assumed above, which we have supposed to remain constant as ${}_2X$ and ${}_3X$ approach ${}_1X$. Also, since $(d)dx$, (since it bears an assignable ratio to $(d)x$, however nearly ${}_2X$ approaches to ${}_1X$, or

[74] This makes $(d)dx$ an inassignable. It may be a misprint due to a slip of Leibniz, or of Gerhardt in transcription; for there is no similarity between it and the statement in the next line. I cannot however offer any feasible suggestion for correction.

[75] This is quite wrong. Leibniz has evidently substituted $x+dx$ for x, etc.; which is not legitimate unless ${}_3X_3Y$ is taken as $y+dy+d(y+dy)$, and so on; even then fresh difficulties would be introduced. As it stands, this line should read

$$a\,dy + a\,ddy = x(dv+ddv) + v(dx+ddx) + (dx+ddx)(dv+ddv).$$

On account of this error and that noted above, there is not much profit in considering the remainder of this passage.

however much dx, the difference between the abscissae, is diminished), is not evanescent, even when, finally, dx and ddx, dv and ddv, are all supposed to be zero. In the same way, the ratio of ddv to ddx may be expressed by the ratio of an assignable straight line $(d)dv$ to the assumed constant $(d)x$; and even the ratio of $dv\,dx$ to $a\,ddx$ may be so expressed; for, since $dv : dx = (d)v : (d)x$, therefore $dv\,dx : dx\,dx = (d)v : (d)x$. Hence, if a new straight line, $(dd)x$, is assumed to be such that $a\,ddx : dx\,dx = (dd)x : (d)x$, then the new straight line will be assignable, even though dx, ddx, etc. become evanescent. Since therefore $dv\,dx : dx\,dx = (d)v : (d)x$ and $dv\,dx : a\,ddx = (d)x : (dd)x$, it follows that $dv\,dx : a\,ddx = (d)v : (dd)x$, an thus at length there is prouced an equation that is freed as far as possible from those ratios that might become evenescent, namely,

$$\frac{(d)\,dy}{(d)dx} = \frac{x}{a}\frac{(d)\,dy}{(d)dx} + \frac{y}{a} + \frac{2\,(d)y}{(dd)x} + \frac{2\,dv}{a} + \frac{2\,dx}{a}\frac{(d)\,dy}{(d)dx} + \frac{ddv}{a}.$$

Thus far all the straight lines have been considered to be assignable so long as ${}_1\mathrm{X}$ and ${}_2\mathrm{X}$ do not coincide; but in the case of coincidence, dv and ddv are zero, and we have

$$\frac{(d)dy}{(d)dx} = \frac{x}{a}\frac{(d)\,dv}{(d)dx} + \frac{v}{a} + \frac{2\,(d)y}{(dd)x} + \frac{0}{a} + \frac{2\,(d)dv}{(d)dx}\frac{0}{a} + \frac{0}{a},$$

or, omitting terms equal to zero,

$$\frac{(d)dy}{(d)dx} = \frac{x}{a}\frac{(d)\,dv}{(d)dx} + \frac{v}{a} + \frac{2\,(d)y}{(dd)x}.$$

Hence, if dx, ddx, dv, ddv, dy, ddy, are by a certain fiction imagined to remain, even when they become evanescent, as if they were infinitely small quantities (and in this there is no danger, since the whole matter can be always referred back to assignable quantities), then we have in the case of coincidence of the point ${}_1\mathrm{X}$ and ${}_2\mathrm{X}$ the equation

$$\frac{ddv}{ddx} = \frac{x}{a}\frac{ddy}{ddx} + \frac{v}{a} + \frac{2}{a}\frac{dx\,dy}{ddx}.$$

J. M. CHILD.

DERBY, ENGLAND.

LIBRA:

THE ETERNAL BALANCE OF GOOD AND ILL.

I.

FROM everlasting is the Universe,
And unto everlasting shall extend;
Without beginning is it; without end
Its morrows ever yesterdays rehearse;
Not first nor last but only midst it knows;
As never young, so never old it grows.

II.

Yet is the secret of its permanence
Not rest but striving, not a dead repose,
No peace of mutually slaughtered foes,
Nor truce of wearied, but a strife intense,
Deathless, of powers that charge and countercharge
Ever, yet never may their bounds enlarge.

III.

Not progress is the secret of the sky,
And not decay the withering doom of earth;
Though, out of star-mist, systems round to birth,
And a dead moon mirrors earth's destiny,
The star shall sink in darkness whence it came,
And earth's grim desert be reborn in flame.

IV.

It is the wave with endless rise and fall,

It is the tide with ceaseless ebb and flow,
The changing moons, the hours with gloom and glow,
That hold the mystery of each and all,—
The rhythmic secret, wherein man has part
Even from the first pulsation of his heart.

V.

The pendulum with its untiring swing
Not only metes out time, but it reveals,
Babbling, the word eternity conceals,
Though to men deaf with their own questioning;
The lilting ripple of the poet's song
Itself contains the clue he sought life-long.

VI.

Nothing can be unfolded but has first
Been folded in, and shall be so again;
Nor yet can aught in equipoise remain,
But ever driveth toward the best or worst;
Nature keeps neither full nor empty cup,
And the half-filled she drains or fills it up.

VII.

Yet what had no beginning always is
And never can become; no inward change,
However wide its outward motions range,
Can touch its heart; despite man's fantasies,
The Universe exists, not merely seems
An everlasting see-saw of extremes.

VIII.

These two extremes man knows as More and Less,
As Good and Ill, lastly as Right and Wrong;
Feels them as Love or Hate his pulses throng;

Sees them with Beauty clothed or Ugliness,
And names them from their power to bless or ban
God and Devil, Ormuzd and Ahriman.

IX.

The righteous Paul lamented in his heart
The Good by Evil thwarted. So in thine
The False and True, the Cruel and Benign,
The Pure and Impure make thee what thou art
And what the All is: tiger, dove, and man,
Seraph and fiend, are fashioned on one plan.

X.

Even as the Universe, mid seeming change,
Really is locked in iron permanence,
So, everywhere, despite our cheated sense,
From one self-nature may it never range:
One is it, one in body and the soul,
And every part is parcel of the whole.

XI.

Behind all forces hides the primal Force,
The Unconditioned, which is bad and good
Impartially, and its divided mood
The single spirit of the Universe,
Of you and me and all men and the earth
And all the worlds Infinity wheels forth.

XII.

But mortal life displays not one but two,
Shows Good all-perfect warring against Ill,
Which yet abides unconquerable still,
And in this duel sets for man a part,
And teaches he must choose the side of Good,
Or rank below the cleft, insensate wood.

UMI

XIII.

Had it been destined to be otherwise,
Long since it would have been so; nay, for we
Deal not with time but with eternity,
It would have been so always; had our skies
Been fated to o'erarch a perfect earth,
They would have overarched it from their birth.

XIV.

This is the revelation; this alone
Rained ever from the Milky Way adown,
Or flamed from Vega and the Northern Crown,
Even this that written on my heart I own.
Not ours to ask if unto me or you
The word be welcome, but if it be true.

XV.

What then must be the Universe, ideal?
Never and nowhere; but endurable,
A place where on the whole 'tis fairly well,
Where at least men can live; in short, the real.
Had it been more, there were no need to ask;
Had it been less, not ours had been the task.

XVI.

If this be true, as Life forbids to doubt,
Is low then one with high, is conscience vain?
Forever no! But, though I shall not gain
After short strife a glorious mustering out,
My privilege more glorious is to be
A soldier of the Right eternally.

XVII.

Yet what avails my battle for the Right,

You ask, if through eternity shall still
Be kept the balance between Good and Ill?
Me much avails it, for 'tis mine to fight
On the Lord's side, being birthmarked with his seal;
My joy, my life is in that battle peal.

XVIII.

More can I ask? Shall some far eon see
The Evil quelled, the Good supreme prevail?
Not if our world have told us a true tale.
But can we hear and judge it rightfully?
Our torch is feeble; but at least its light
Reveals us friend and foeman in the fight.

XIX.

The rest is God's. Yet who would change that could
Doom so divine, which loftiest souls must bear,
Though archangelic?—in all worlds to share
The warfare of the soldiers of the Good,
Though marching under orders ever sealed,
And battling ever on a doubtful field!

HARRY LYMAN KOOPMAN.

PROVIDENCE, R. I.

CRITICISMS AND DISCUSSIONS.

LOGIC AND PSYCHOLOGY.

The nature and purpose of symbolic or mathematical logic, which began to be developed by Leibniz and was continued quite independently by Boole and others, is tolerably well known by now. Logical reasoning is translated by it into what Leibniz called a "real characteristic" which is very analogous to ordinary algebra, and helps swiftness and accuracy of reasoning—even complicated reasoning—in much the same way as the signs in algebra do. This tendency culminated in the very ingenious and useful "mathematical logic" of Peano. Peano's system was far more complete than Boole's, for the whole of a piece of reasoning which included algebraic formulas and equations could be put into a symbolical form in which ordinary words—which are not part of a "real characteristic"—are not used. In this direction Peano's system met the much earlier system devised by Frege. However, Frege's system was not thought out so much with a view to rapidity of reasoning and convenience of writing as with a view to emphasizing slight and important logical distinctions in very similar concepts and deductions and consequently a scrupulous accuracy in deductions. It may thus be noticed, by the way, that the purpose of Frege's symbolism was different from that of all previous symbolisms in logic and mathematics, for Frege wished to lay stress upon the *differences* in various analogous ideas and deductions rather than upon their *analogies*. Broadly speaking, Russell and Whitehead's work may be characterized by saying that it is formed under the influence of a combination of the two tendencies represented by Frege and Peano. The convenient symbolism of Peano is retained wherever possible and the superior analysis and subtlety of Frege is fully used. We ought to add also that nearly all of Frege's dis-

UMI

coveries were made independently by Russell himself, Frege's great work having been neglected by philosophers and mathematicians.

There is one great point in which Russell's works differs much from that of Frege: full use is made of the enormously important researches of Georg Cantor on transfinite numbers. While putting on a firm basis the treatment of infinite classes and numbers, Cantor's work led to the recognition of forms of a paradox absolutely fundamental in logic. After many vain attempts by various mathematicians and philosophers, this paradox has been satisfactorily solved by the thorough remoulding of logic given in Whitehead and Russell's *Principia Mathematica.*

From Peano's various *Formulaires* to the work last mentioned the subject-matter is principally the collection of truths which we can reach by logical deduction from logical principles. This body of truths is not a description of psychological methods of discovery or psychological results, but is of course reached by psychical processes, like most other discoveries in a purely intellectual domain. It is then simply irrelevant to complain that there is no place in the *Formulaires* or *Principia* for that "intuition" which brings about mathematical discoveries. It would be just as much to the point to complain that in what is excavated we do not discover the tools used for excavating or the method of excavation. And yet this is what the rather superficial and amusing discussions of Henri Poincaré are mostly about. And these discussions are what Prof. J. B. Shaw in the number of *The Monist* for July, 1916, refers to (p. 397) as Poincaré's "successful attacks on logistic." We might reasonably, it seems to me, have expected that Professor Shaw should make some reference to the reply by Louis Couturat to Poincaré which was translated in *The Monist* for October, 1912, and which is quite conclusive on so many points. Professor Shaw, in his eloquent and somewhat inaccurate (both from the points of view of history and logic) attack on mathematical logic, urges what are, at bottom, the very same irrelevant arguments. I shall try to point out some of these inaccuracies, both because they are fairly common even now among mathematicians, and because it is surely the duty of every one to contribute as far as he can to the clarification of notions in America above all other countries; for it is from America that we expect an exceedingly large proportion of the work of the intellect in future now that Europe has deliberately handicapped herself.

UMI

Professor Shaw's slighting remark on the impotence and boasting power of logistic (p. 411) is the result of a strange misconception. Logistic deals with logical entities and deductions which are fundamental to mathematics, and it is unjust to try to make people believe that logistic ever claimed to be the overlord of mathematics. There seems, in fact, to be a note almost of personal dislike for logistic in those mathematicians who attack it. And yet the question is wholly concerned with logical facts, and is not to be answered by rhetorical appeals to prejudice or sentiment. If logic is more fundamental than mathematics, why should there be any objection to the—successful as it happens—attempt to define mathematical entities in terms of logical ones? If mathematics is more fundamental than logic, the first thing to do is to draw up a scheme showing that logical entities can be deduced from specifically mathematical ones. Until this is done, and certain objections to it are at once obvious, it is quite unconvincing to disparage cultivators of logistic. After all, logisticians are working at mathematics in much the same way that other mathematicians are. They are concerned with more fundamental problems and problems which do not so easily appeal to the public, as, say, a proof of Fermat's great theorem would, but they discover truths just as much as any other mathematicians. They introduce conceptions to work with. We may mention the idea of *propositional function* actually mentioned by Professor Shaw in terms of commendation (p. 411), which was introduced implicitly by Boole and MacColl—both early mathematical logicians—and explicitly by Frege, Peano and Russell—all logisticians. A small acquaintance with such a work as that of Frege will give plenty of examples of other powerful new ideas introduced. And then as to truths discovered by logisticians, we may remind Professor Shaw that the solution of "the paradoxes of logic" is wholly due to them, while mathematicians who were unacquainted with logistic hopelessly floundered in the search for a solution. Twelve years ago I was one of these flounderers myself, and my "solution" had been accepted as satisfactory by many mathematicians.

The real fact is that these results of logistic do not strike some mathematicians as nearly so important as some of the results of the theory of functions, for instance. I think they forget that it is only in virtue of all truths being really of equal "nobility" that Jacobi was right in claiming that a theorem in the theory of

numbers was just as fine as a very striking result in mathematical astronomy.

Perhaps the greatest mistake made by Professor Shaw is the extraordinary statement about the nature of truth near the top of page 409. It is surely quite evident that truths themselves do not develop. That twice two are four was just as true last year as it will be next year,—even if no people at all are left alive on the earth next year. Professor Shaw finds fault with something I wrote because he thinks that I maintained that ideas are not created by man. It is quite evident from what I said in the context that I only held that *truths* are not created, though I certainly said in a slipshod and inaccurate way that "we do not really create *anything* in science." Really Professor Shaw shows afterward that he agrees with me that truth itself is not created, and his remark that doubtless I thought that words and ideas waited in the mines of thought for the lucky prospector does not appear to be either logical or a good guess (see pp. 409-411). However, at the top of page 409 he remarks that the world of universals changes in time. I suppose that he means that our ideas, say of an "integral" or "continuity" have changed; but I hardly think that he ought to have fallen into the error of mistaking the thing itself for a result of our groping after the thing. I take it also that he does not intend to say that truth evolves, for that rests on a confusion between a *proposition* and a *propositional function*, such as in thinking that such a function as "Dr. Wilson is President of the United States" is a proposition and not a function of the time which becomes a proposition when any instant is specified and is then constantly true or false eternally. What is the case seems to me to be that in logic and mathematics the world we are concerned with is a world of facts, not of conceptions. *Conceptions* are formed by us for the purpose of stating truths, and in the world of pure mathematics we only come across facts and *form* and *variables*. In this I think that I shall have the support of one at least among philosophers: I refer to Dr. Carus, who has always maintained that mathematics is essentially concerned with the ideas of *form* and *"anyness."*

We now come to the last inaccuracy in Professor Shaw's paper that I shall deal with. This is the question about the logic of infinity. The inaccuracy of the statements on page 412 appears clearly if we give a short statement of the facts in the treatment of infinity by mathematicians and logicians. Georg Cantor, in a

UMI

series of works dating from 1871 to 1897, succeeded in founding a new and immensely important theory of transfinite numbers. The use of the lowest transfinite cardinal numbers did not and does not present any difficulty whatever to mathematicians or even logicians; but, as Burali-Forti, Russell, and others noticed in various forms the *whole* series of transfinite numbers presents difficulties which were later found to be fundamental logical difficulties of the same nature as that of the Cretan who said that all Cretans were liars. Such problems were discussed at length in Russell's *Principles of Mathematics* of 1903 and in the years after the publication of this book were satisfactorily solved by him and Whitehead. These solutions may be found in the *Principia* of 1910, and in the almost wholly symbolical form of the book last mentioned it is naturally impossible, even if it were not superfluous, that the claims made in the earlier work should be repeated. Thus it is unjust to conclude (p. 404) that the *Principia* is an abandonment of the claims of the *Principles*, brought about because of the difficulties found in Cantor's work. One might just as well conclude that the difficulties of a solution of the great difficulty of "Cantorism" had made Russell give up joking, for there are many jokes in the *Principles* and only one in the *Principia*. There is one more point. It is only what we may call a "boundary problem" about Cantor's numbers that gives rise to difficulty: the resolve that any object about which we talk or reason must be defined in a finite number of words (p. 413) does not succeed in putting out of court all classes that have an infinite number of members. Infinite classes of objects each of which can be finitely defined can be defined in a finite number of words, or better symbols of a "real characteristic." The class of prime numbers is such a class. If indeed we may use the notion of *any* (which is represented by one word) or the notion of a *variable* in general, we cannot avoid admitting definitions of infinite classes by a definite number of words. If also we may use a sign for a variable, there is no earthly difficulty in giving a general rule for correspondence in a way that is denied by Professor Shaw on page 413. The rule, for example, if n is an integer, given in the formula $n+p$, where p is another integer, indicates precisely another class of integers which is correlated to the whole class of integers considered first.

There is a small logical error committed by Professor Shaw, at least if he considers philosophy to be the same thing as meta-

physics, which may explain why he is so satisfied with himself for ignoring philosophy. On page 409 he characterizes a certain assumption as "philosophical" and explicitly divides "philosophy" from "mathematics." On page 414 he agrees with Lord Kelvin that "mathematics is the only true metaphysics." Thus he would seem to hold that there is no such thing as philosophy at all; this would certainly explain why philosophical assumptions are so little worth serious discussion. Such discussion would in fact be as foolish a problem as to investigate the birthplace of Jack the Giant Killer's hen. But if we try to take a somewhat broader view, and are not satisfied with dividing our knowledge into arbitrary watertight compartments labeled "Philosophy," "Mathematics," and so on, we see that there are certain logical questions which can be and have been solved by symbolical methods which strongly remind us of algebra, which are absolutely fundamental in mathematics, and which when formulated in ordinary language sound so like what professional philosophers have often talked about that many are tempted to hurry them out of sight into the "philosophical" compartment. These are some of the questions with which logistic deals. Logistic never claimed to be able to run without the guidance of a human intellect (see p. 411) any more than the sciences of mathematics or logic or chemistry did. What it does claim to do is, like ordinary mathematics, to save our minds the labor of performing again each elementary reasoning which requires no talent but only memory—often a prodigious memory when the reasoning is complicated; so that we can reserve all the talents we may possess for overcoming those obstacles to a discovery of truth that have not been hitherto overcome. Then again, unlike ordinary mathematics, logistic seeks to point out differences in analogous ideas and reasonings which play an even greater part then analogies when we come to consider really subtle reasoning. Thus the analogy between implication between propositions, inclusion between classes, and inclusion between relations breaks down in certain cases, and we see that Russell in his later work forsakes the identical form of the symbols expressing these relations. Peano, as we know, kept to the same symbol on account of the very close analogy between the relations spoken of.

If we are content to accept without examination the arbitrary classification of people who were unacquainted with modern logic into exclusive "mathematical" and "philosophical" compartments,

UMI

we must be prepared to think we see what we think are the rigid foundations of mathematics being eaten into by philosophy, and if we wish still to maintain that the foundations of mathematics are rigid we shall have continually to give the new name of "philosophy" to parts of what were hitherto considered to be mathematical. This state of things was actually brought about by what Poincaré called "Cantorism": truths which were hitherto considered solid and mathematical seemed to be thrown into doubt by the advance of philosophy. Of course this was not really so: the logical questions at the foundation of mathematics are capable of scientific investigation just as much as the theory of numbers of the differential calculus, and it is unnecessary and ridiculous to narrow the scope of our investigations because we shall meet logical difficulties if we do not. What would be thought of a tradesman who thought he could calm the mind of his assistants by maintaining that the ravages of a bull, although they seemed to be in his own china shop, were really in a drapery department which had somehow extended into that part of his shop where plates were sold? This is what those mathematicians do who dismiss awkwardness to "philosophy" and think that thereby they have kept mathematics pure and free from all "metaphysical" discussions.

Miss Dorothy Wrinch has sent the following comments on Professor Shaw's article in *The Monist* for July, 1916:

"The chief thing that I quarrel with in Professor Shaw's article is his idea of *one*: selecting one pencil from a pile is really rather different from considering the class whose members are the classes 'living kings of England,' 'fathers of A,' etc. Further, it would be difficult to give a definition of *one* or *two* which is not a statement in which one or two appears: he does not attempt to say that the other constituents of this 'statement' have not been defined, or that the definiendum is not unique. These could be his only grounds for attacking a definition, which is merely a statement in symbolic form of cases in which the number one or the number two appears. Also it seems a pity (line 9, p. 407) that he should fall into the error that he deplores in mathematical logicians, viz., the error of introducing the notion of truth (and truth value) when 'in no place.... they are defined.'

"I suppose that it was 'in the intoxication of the moment' that Professor Shaw called a propositional function of two variables a relation (p. 404, line 14), and let out of the bag the existence of

a difference—hitherto, apparently, kept dark by mathematicians—between the properties of the roots of a quadratic equation and the properties of quadratic functions of x.

"Professor Shaw makes some strange remarks on page 413. If the collection of all integers does not exist (line 8) it seems hardly necessary to refute the proposition that it is possible to correlate the collection of all integers to some other infinite collections.

"It seems rather unsportsmanlike to rely upon people's short memories and call Poincaré's attacks on logistic successful. Might it not be well to remind people of the conclusions to which M. Couturat came at the end of his article in *The Monist* for October, 1912: 'Admitting the principles and primitive ideas of the logisticians, M. Poincaré has maintained that, setting out from these data, they cannot build up mathematics without another postulate—an appeal to intuition or a synthetic *a priori* judgment; and he has thought that he has discovered in their *logical* construction certain paralogisms (beggings of the question or vicious circles). I believe that I can conclude from the above discussion that not one of these theses is proved, and that, in particular, the logisticians have not committed any of the logical errors that are so lightly imputed to them.'"

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THE CAL-DIF-FLUK SAGA.

EDITORIAL INTRODUCTION.

Mr. J. M. Child has given, in the following "Saga," an amusing description of the results he has arrived at in his book on Barrow, just published in the series of "Open Court Classics." The closing lines represent the opinion he has formed from a consideration of the manuscripts of Leibniz, an annotated translation of which has been appearing in current numbers of *The Monist*, beginning with October, 1916, and continued in the April number and the present one.

The saga evidently refers to the question of the invention of the infinitesimal calculus. Isa-Roba is Barrow, Isa-Tonu is Newton, Zin-Bli is Leibniz, while Cavalieri is mentioned under the name of Ler-a-Cav. Gen-Tan-Agg stands for Barrow's *Gen*-eral method of *Tan*-gents and of *Agg*-regates; while Shun-Fluk and Cal-Dif ob-

viously refer to the methods of Newton and Leibniz. Batnac is the ordinary abbreviation of the Latin for Cambridge, Cantab., with its letters reversed; and the allusion in the next line is to Barnwell Pool, where it is stated that an undergraduate whose boat had overturned was saved from drowning, but died soon afterward from blood-poisoning! Terangel is a transformation of Angleterre, i. e., England. Ris-Pah is Paris, where Leibniz lived at the time of the invention of the calculus.

In the second stanza, the allusions to "burning midnight oil," the quill pen, incandescent gas mantle, and the electric light are all fairly obvious; while the Swan may be taken to refer to a well-known make of fountain pen. Stanza 5 refers to the publication of a book. The archery in the first method of training alludes to the ancient definitions of a tangent and a normal to a curve; and the sword-play recalls *Euc.* I, 10 and *Euc.* I, 1, while the allusions in the second are easily referred to the method of indivisibles.

In Stanza 9, the dagger refers to the differential triangle, which Barrow only included in the first edition of his work on the advice of Newton; the knobs on the hand-grip refer to Newton's "dot" notation.

The two weapons of Zin-Bli are the signs invented by him for differentiation and integration. Lastly, Li-Nu-Ber is John Bernoulli, who stated that Leibniz got the whole of his fundamental ideas from Barrow, whereas Leibniz himself denied any indebtedness to Barrow.

THE CAL-DIF-FLUK SAGA.[1]

1. Saga of sons of a Goddess, of Thought and Learning the fountain,
(Haply in that which I sing, a real historical meaning,
Wrapped in a fanciful garb, and oddly disguised as a saga,
Those who are skilled in lore, and erudite more than their fellows,
Knowing the facts of the case, if they diligently seek may discover.)
Dwelt She, She dwells upon Earth, and henceforth for ever and ever
Dwell so She will among mortals. 'Tis thus decreed by the All-Wise.

2. Oil from the Midnight Lamp the sacrifice burned on her altars,
Plumes from the wing of the Goose her now peculiar token;
Not so at first was it thus, and not in the times that are coming

[1] From a manuscript found in 1916 A. D., while searching an ancient tumulus or "barrow," and made out from the original by J. M. Child.

Will it be Oil and Plume; I see with the eye of the seer
Wondrous visions of Light, enwrapped in a Mantle resplendent,
Torn from the heart of a stone, the essential soul of the Sun-god
Prisoned for ages therein; and globes of crystal translucent
Glowing with filaments bright kept hot by the Spirit of Lightning,
Swan of the Golden Beak instead of the Goose for her token.

3. Sent upon Earth to dwell with mortals by will of the All-Wise,
Children divine to bear to those who Her fancy might capture.
Ardent and long was the wooing, both strong and patient the lover,
Ere he received his reward, or ere She presented him offspring.
Else as a mark of Her love to him She had chosen to honor,—
Chosen for womanly whim, for some unaccountable reason
Honored above all else, who never had courted her favor—
Sent She on lighting wings the soul of Her heart, Inspiration.

4. Children of fathers of Earth, but endowed with the life of the Mother,
Destined as Heroes to wage perpetual warfare on all things
Troubling the minds of men desiring to widen the limits
Set on the realm of We-Know, by the race of the children gigantic,
Issue of Never-Before out of We-Never-Heard-of the-Method.
Children begotten from Her are known by the names of their fathers,
More by the deeds of the sons are the fathers so held up to honor;
Accurate records are kept; thus long through the ages that follow,
Known by the deeds of the sons are the fathers so held in remembrance.
Rightly was this the Law, for responsible he for the training,
Fitting the son for the fight for freedom and fuller perception.

5. Till 'twas such time as was meet, the custom obtained that in secret
(Jealous that others might see not fully developed the power
Promising greatness to come), this fatherly training continued
Day after day for an æon; until with a flourish of trumpets,
Front of the eyes of all, tattooed with the symbols of Learning,
Clad in a mantle of calf-skin, bearing on back and on bosom
Plainly for all to observe, in resplendent gold letters, his title,
Son of the Goddess of Thought, was he set as a champion of Knowledge.

6. The methods of training were two, at least only two were accounted.
Oldest and best known of all was the method derived from the Ancients,
Cumbrous, exhaustive and long; horizontal and parallel bar work,
Drawing of cord of the bow, and the rings were considered essential;
Accurate hand and eye were developed by shooting an arrow,
Grazing the cheek of a figure, or forth from it standing erected;
Cleaving a bar into twain, so each part as to balance the other
(Nought but two measuring swings ere the cut was delivered allowed him).
Such like in days of old had fitted the Heroes for battle.
Founded on this was the second, but strangely unlike it in practice;
Suppleness rather than strength was the object and creed of the trainer.
Straight-edged still was the sword; with it blocks were sliced into shavings,
Shavings were sliced into threads, and threads were chopped into pieces,
Parts of ineffable smallness, divisible reckoned no further.
Masonry part of the course, in which arches with bricks were fashioned,
Leaving the corners undressed; as the pupil advanced in his training,
Smaller and smaller the bricks, indivisible finally counted.
Specially fitted for Heroes, prepared for attack on the giant
Clans of A-Re-A and Vol-Yum, the brood of Cur-Va-Rum and Mez-zur.
Failed if the fatherly training, the Goddess in sorrowful anger
Took from the child his soul, the gift which at birth She had given,
Worthier father to bless, if ever another such won Her.

7. Once in the days now gone, there lived on the banks of the Batnac,
Renowned for its smells and its mud, where pollution enters at Well-Barn
(Truly not then was this fame, nor yet at the time of this writing
Thus had it won a repute, 'tis a prophecy sure that I utter),
Land of Terangel within, a mortal yclept Isa-Roba.
Many and varied his loves, his fickleness surely a drawback;
Truly a wonder it was that the Goddess e'er let him approach Her.
Bare She however a son, Isa-Roba undoubted the father,
Fair both in face and in form, a divine conception befitting;

Ne'er such a babe before was born with so splendid a future;
Seemed that the soul of his Mother had enter'd the Child at his birth-time:
Best that She had to give, best that She can give for all time,
Gave She this son of Her heart; Gen-Tan-Agg Isa-Roba did name it.

8. Trained he the boy in a manner that savored of that of the Ancients,
Discipline rigorous keeping, yet toned with a method that fore-time
Ler-A-Cav brought to perfection, a mingling of first and of second
Systems of training recounted; however 'twas foredoomed to failure.

9. Hercules never so strong as the youth Gen-Tan-Agg, no, nor Samson.
Armed with his two-handed weapon he met many giants in combat;
Numerous clans he defeated, by slaying their general doughty.
Nevertheless were his muscles too stiffened by reason of rigor,
Due to the manner in which Isa-Roba conducted his training.
Love for the two-handed broadsword, with which Isa-Roba had armed him,
Made him neglect the superior weapon that hung at his waist-belt,
Sharper by far than the sword-blade, a steel of superior temper;
Seems Gen-Tan-Agg only used it preparing the shafts of his arrows;
Nigh came to leave it at home as he set out upon his first journey,
Girding it on at the last, not perceiving in it that a weapon
Ready to hand he had got against which no armor of mortals
Could for a moment prevail; for piercing the joints of the harness,
Off'ring no passage to sword-blade, it reached his opponent's main vitals;
Forced him to give up his treasure, the secret protected for ages.

10. Happened it thus that a Hero, high-blessed by the Goddess his Mother,
Spoiled by the weapon mistaken his anxious sire recommended,
Fame and renown and great honor did miss for ever and all time,
Losing the chance that was offered, a name and a high reputation.
Lastly, by father discarded (who fickly returned to a first love),
Languished and nigh came to perish, unhonored, unsung and neglected.

11. Some of the records of giants the youth Gen-Tan-Agg had defeated

Chanced Isa-Roba, however, had told to a friend Isa-Tonu;
Agile by nature, the latter immediate saw that the dagger,
Superior far to the broadsword, was a weapon of magical value;
('Twas Isa-Tonu's advising that just at the very last moment
Caused Isa-Roba to add to his offspring's armor the dagger).
Pity, perhaps, for the youth, or a covetous eye for the poignard,
Caused Isa-Tonu to take neath his fostering care the young stripling,
Freeing his father from trouble, unhampered to follow his fancy.
Thus Isa-Roba the story departs from, unhonored for all time;
Save and if only in future, this tomb may be opened by some one
Trying to find out the truth of the Hero's father and birthtime.
Under the fostering care of a trainer less hide-bound by nature,
Slowly at first, then apace, did the Hero recover his power.
Changed was his armor, the sword altogether replaced by the dagger,
Changed was the dagger in form, for a knob, sometimes two, on the hand-grip
Gave it a far better balance. Obsessed by his special requirements,
Secretly long Isa-Tonu did bind Gen-Tan-Agg to his service.
Later ungratefully hiding the name of the Hero who served him,
Swearing that all had been done by his own bastard offspring, young Shun-Fluk.

12. Thus once again was the Hero discarded and left for to languish,
Shun-Fluk attaining the fame that should his have been truly and rightly.
Nemesis, son of old Equity, sternest of Gods and the justest,
Saw Isa-Tonu's deception, and straightway the Goddess of Learning
Sought He and told Her the story. In sorrowful anger the Goddess
Listened with eyes that flamed at the failure that followed Her offspring,
Due to his father's bad training, and then Isa-Tonu's enslavement;
Listened and cursed the first, for the other a punishment thought out.

13. "Punishment dreadful and dire!" So she spake, the while Nemesis listened,
Listened and nodded and smiled, as approved He the plan She suggested.
"Lives there a mortal in Ris-Pah, who long has courted my favor;
Often of late have I thought that at last I'd reward his devotion.
Lacks he but one little thing, only one thing to render him fitting

Trainer of offspring of mine; but the lack mean I now to forgive him.
Never again could I bear such a child as I bore Isa-Roba;
Certain is that; but immortal the soul that at birth-time I gave him,
Breath of my life, Inspiration, again, Gen-Tan-Agg expiring,
Can, if I will it, enlighten the child which I'll offer to Zin-Bli.
Thus is he called by mortals, an inventor of weapons and symbols.
One has he fashioned already, in shape like a chopper for fire-wood,
Straight in the shaft, with a hand-stop to stay it from slipping,
Circular edge to the axe-blade, to shaft is it fastened by bolt-head;
Much like the symbol that mortals set fourth in the lower-case system.
This shall he teach my offspring to use to more delicate purpose.
Binds he his sticks all together with cord made out of the sum-omn;
Lurking however in thought is the germ of a better invention,
Rod with curl at each end, slightly bent, so that clipped round the bundle,
Binding the whole into one, he is able to thus grasp it firmly.
Armed with each of these twain, shall his offspring forth stand as a Hero."
Spake She, and Nemesis nodding to all His approval, it was so.

14. Cal-Dif named Zin-Bli the child, and he trained him these weapons to master;
Speed, at all rates, with the first he created new records completely,
Nor did he stay at that; with the second, the brood of the giants,
Laid he them low in the dust, so that never again should they trouble.
All that the Goddess had said, so performed She; the credit of Cal-Dif
Famed through the kingdoms of mortals, became a renown for the father,
Ne'er to be equalled till Earth is devoid of reasoning mankind.

15. Swelled as to head by renown, though Zin-Bli well knew Inspiration
(Could he forget this?) had wrought in a magical manner the marvel,
Yet could not bear it for others to know whence the source of his wisdom;
Denied he the source whence it came, Isa-Roba's offspring discarded.
Nemesis saw what he did, and he stirred up the folk of Terangel,

Shun-Fluk to accuse him of stealing and sending him forth as his
Dif-Cal.
None seemed to have guessed the truth, save a man by the name of
Li-Nu-Ber.

16. Ye who perchance may consider this saga in future far ages,
Know now the truth ye may; that the soul of the Goddess of Learning
Entered at first Gen-Tan-Agg, but he languished for lack of good
training;
Afterwards, renamed Shun-Fluk, he recovered some of his birthright;
Dying, his soul was then given to an ordinary child of a mortal,
Rendering its face and its form like one of divine conception.

17. Accepted as such by all, till the day that this saga's discovered,
Haply e'en then, for foretell I that Cal-Dif.....

Unfortunately, the manuscript, which consists of another couple of sheets that were outermost in the roll, here becomes indecipherable through being destroyed by damp; it would have been interesting, and useful in the light of judging of the truth of the facts given, to have verified how far the prophecies were fulfilled by events since the time at which they were written down and the manuscript hidden in this old burial-mound.

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NOTES ON DE MORGAN'S BUDGET OF PARADOXES.

In a work requiring the large amount of reading involved in editing a book like the *Budget of Paradoxes,* and particularly in the condensing of the results to the proper proportions for footnotes to aid the reader, it was, of course, inevitable that a certain number of inaccuracies would occur. It is also evident that many more notes might profitably have been added to elucidate the meaning of the text, or to correct the original where this would be warranted.

De Morgan was a careless writer and many of his errors are mentioned in the footnotes; but numerous others exist, some of which are patent to any reader and others of which might profitably

have been set forth by the editor. It is also a serious question as to whether the translation of common phrases is not more of a hindrance than a help to even the casual reader, and whether the space used by such translation might not have been more profitably devoted to a further elucidation of the text. This is the feeling of one or two critics.

Since the work was published, several friends have called attention to a few misprints, a few generous critics have suggested helpful changes, and one or two others have objected to certain of the notes. It therefore seems proper to present a few emendata and errata which may assist the reader of the work.

In the matter of emendata to De Morgan's text itself and of suggestions as to further helpful notes I am indebted chiefly to Prof. A. E. Taylor of St. Andrews, Scotland, who has gone over the work with great care and has kindly given the Open Court Publishing Company the benefit of his reading. The following notes on De Morgan's text are due to him.

Vol. I, page 3. De Morgan should not have attributed to Spinoza the anonymous *Philosophia sanctae scripturae interpres.* It was probably the work of his friend and physician Lodowick Meyer.

Vol. I, page 41. De Morgan's version of the passage from the commentary of Eutocius on the tract by Archimedes on the measure of the circle is not satisfactory. The Cerii of Porus should be the Ceria (*κηρία*, honey combs) of Sporus. He probably used the Wallis edition of Eutocius and quoted only the first four words of the passage (*Archimedis Opera Omnia*, III, p. 300, of the 1881 edition of Heiberg): *εἰς ἀκριβεστέρους ἀριθμοὺς ἀγαγεῖν τῶν ὑπ' Ἀρχιμήδους εἰρημένων, τοῦ τε ζ″ φημὶ καὶ τῶν ι′ οα″*. The restoration adopted by Heiberg makes the statement of Eutocius correct: "a more accurate evaluation than that of Archimedes, i. e., than the fractions $\frac{1}{7}$ and $\frac{10}{71}$." According to Sporus, Philo of Gadara had found closer limits. Archimedes had given $3\frac{1}{7}$ as the upper limit and $3\frac{10}{71}$ as the lower limit of π, the ζ'' and $o\alpha''$ representing merely the fractional parts.

Vol. I, page 96. De Morgan's language seems to imply that the Convocation of the University of Oxford is, or was, a body of ecclesiastics of the Anglican Church, but it is not an ecclesiastical body at all. It consists of all masters of arts who qualify by the regular payment of their university dues. Professor Taylor suspects that De Morgan may have confused the Convocation of Oxford with the Convocation of the Clergy of the Province of Canterbury.

UMI

Vol. II, page 274. For De Morgan's translation of οὖλον μέλος, read "a song of bale" (ὀλοὸν μέλος).

Vol. II, page 277. De Morgan overlooks the true reason why Pope scans *Mathesis* as *Máthesis,* namely, that like all writers of his day he pronounced Greek names according to their accent, not as we now do with an adjustment of the stress accent to the quantity of the vowels.

Vol. II, page 322. De Morgan is incorrect in his statement as to Böhme's division of Mercurius. Böhme divides it Mer-cu-ri-us, not Merc-u-ri-us.

Vol. II, page 340. It would be interesting to know whether De Morgan's complaint that Walter Scott did not know what "Napier's bones" were is well founded.

Professor Taylor suggests various other interesting notes relating to the text, and of course such a list could easily be extended.

In the extensive bibliography given in the notes it was inevitable that certain slips of the pen should have occurred. In Vol. I, page 105, I followed Bierens de Haan in giving the spelling "Johannem Pellum." My friend Herr Enestrom has a copy of the edition in question and the spelling there given is "Ioannem Pellivm." He also calls my attention to the proof given in the *Bibliotheca Mathematica* recently that Mydorge was not the author of the *Récréations mathématiques* as published in Boncompagni's *Bullettino.*

Among the slips of the pen which I have noticed since the work appeared is the name of D'Alembert for that of De Lalande in Vol. I, page 41; "condemned" for "contemned" on page 92; and, in Vol. II, "blata" for "beata" on page 61.

Professor Taylor calls attention to the further slips of "fellow of Cambridge" for "fellow of Trinity College, Cambridge" and of "Derion" for "Denon" (Vol. I, page 76); "Viscount of Palmerston" for "Viscount Palmerston" (page 290); "closed" for "classed" (in the text, Vol. II, page 148); "tolo" for "toto" in the text (page 344); and $\pm$ for ± 1 in the text (page 368).

I am also indebted to Professor Taylor for several suggestions of betterment of the translations, matters which should have been attended to by me in the preparation of these particular notes even though I entrusted this work to another. The following changes are not to be attributed to him, although changes (sometimes more extended) were suggested by him.

In Vol. I, page 3, for "what it was" read "that it was"; page 40, for "its appointed path" read "the appointed path"; for the free

translation in verse on pages 53-54, for "And lacking nothing but a start, and lacking nothing but an end," read "The only one without a start, the only one without an end"; page 339, for "think himself to die" read "feel himself dying."

In Vol. II, page 23, n. 4, for "He was wont to indulge in" read "He has a habit of refreshing his reader by"; page 151, for "condemned soul" (literal) read "hack" (colloquial); page 154, change the translation of the familiar legal phrase to bring out the pun upon J. S., "Summum J. S. (for *jus*) *summa injuria*" (the height of law—J. S.—the height of wrong); page 200, change "sleeping power" to "sleep-producing power"; page 228, translate διος εἰμι ἡ ἥρας, as "of Zeus I am, or Hera," and ἡ μίσσα as "mass"; page 260, translate the quotation from Acts xix. 38, as "the courts are sitting"; page 262, for "according to which" read "relatively"; page 283, for the manifest error in the note on *"ab ovo"* read "from the egg," probably relating to the passage in Horace, *"nec gemino bellum Trojanum orditur ab ovo,"* or possibly to *"ab ovo usque ad mala"*; page 365 for "slayst" (misprint for "slayest") read "keepst."

Professor Taylor also suggests that Hobbes lived only about eleven years in France (Vol. I, page 105); that Burnet left England to avoid being involved in the ruin of the Whigs (page 107); that Street acted in accord with the law (page 124); and that there was nothing strange in Laud's patronage of Palmer (page 145). The details of these emendata and certain other suggestions of change would trespass too much upon the space which the editor of *The Monist* has kindly allowed me.

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BOOK REVIEWS AND NOTES.

REFLECTIONS ON VIOLENCE. By *Georges Sorel.* Translated, with an introduction and bibliography, by *T. E. Hulme.* London, George Allen & Unwin, 1916. 7s. 6d. net.

Sorel's book is exceedingly difficult to discuss in a short review. Its substance is a very acute and disillusioned commentary upon nineteenth-century socialism, and upon the politics of the French democracy for the last twenty-five years. It contains also two elements which must not be confused, Sorel's own political propaganda (if he would allow it to be so called) and his philosophy of history formed under the influence of Renan and Bergson. And it expresses that violent and bitter reaction against romanticism which is one of the most interesting phenomena of our time. As an historical document, Sorel's *Reflections* gives, more than any other book that I am acquainted with, an insight into what Henri Gheon calls "our directions."

Doubtless many readers will be disposed to consider the book under its first aspect only. But the study of Sorel's political observations requires an accurate knowledge of government and parliamentary activities since the Dreyfus trial, and does not in itself make the work of importance to the English and American public. What Sorel wants is not a political, but a *social* form. One must remember that his creed does not spring from the sight of wrongs to be redressed, abuses to be cured, liberties to be seized. He hates the middle classes, he hates middle-class democracy and middle-class socialism; but he does not hates these things as a champion of the rights of the people, he hates them as a middle-class intellectual hates. And the proletarian general strike is merely the instrument with which he hopes to destroy these abominations, not a weapon by which the lower classes are to obtain political or economic advantages. His motive forces are ideas and feelings which never occur to the mind of the proletariat, but which are highly characteristic of the present-day intellectual. At the back of his mind is a scepticism which springs from Renan, but which is much more terrible than Renan's. For with Renan and Sainte-Beuve scepticism was still a satisfying point of view, almost an esthetic pose. And for many of the artists of the eighties and nineties the pessimism of decadence fulfilled their craving for an attitude. But the scepticism of the present, the scepticism of Sorel, is a torturing vacuity which has developed the craving for belief.

And thus Sorel, disgusted with modern civilization, hopes "that a new culture might spring from the struggle of the revolutionary trades unions against the em-

UMI

ployers and the state." He sees that new political disturbances will not evoke this culture. He is representative of the present generation, sick with its own knowledge of history, with the dissolving outlines of liberal thought, with humanitarianism. He longs for a narrow, intolerant, *creative* society with sharp divisions. He longs for the pessimistic, classical view. And this longing is healthy. But to realize his desire he must betake himself to very devious ways. His Bergsonian "myth" (the proletarian strike) is not a Utopia but "an expression of a determination to act." The historian knows that man is not rational, that "lofty moral convictions" do not depend upon reasoning but upon a "state of war in which men voluntarily participate and which finds expression in well-defined myths." It is not surprising that Sorel has become a Royalist.

Mr. Hulme is also a contemporary. The footnotes to his introduction should be read.

THE NEW INFINITE AND THE OLD THEOLOGY. By *Cassius J. Keyser*. Yale University Press, New Haven, 1915. Price 75 cents.

In this essay Dr. Keyser shows many interesting ways in which some of the most difficult problems of theology may be partly or wholly overcome by mathematical means.

The relation between religion and science is discussed, the author showing that while science belongs to the middle zone, or rational world, religion belongs to the over-world or superrational. Then follows a brief discussion of the relation of theology to religion, theology being primarily a science, in a word "the science of idealization." From the purely theological standpoint, "God is an hypothesis." In all definitions of God the notion of infinity is foremost. Therefore the essay develops the mathematical concept of infinitudes and through many examples makes clear the denumerable type of infinite manifolds; then far surpassing this in glory, the continuum type, and points to types of even higher orders. "The infinite of theology is the limit of the endless sequence of more and more embracing infinitudes presented by science."

The contradictions of theology are of two kinds, foreign and domestic. Theology may rid herself of the foreign variety by casting out all illegitimate postulates. In the world of infinitudes the part of a group may be just as numerous as the whole group. So in the realm of theology, the seemingly contradictory ideas of omniscience and freedom may be reconciled; for the dignity of omniscience is as great as omniscience itself. The same line of reasoning is applied to the doctrine of the Trinity. The essay closes with a reference to the so-called domestic difficulties, and shows that a being may have many contradictory aspects and yet viewed in a large way all these aspects may be true; just as in comparing different systems of geometry built on various foundations the mathematician finds contradictory facts, yet does not doubt the truth of any of these facts.

Dr. Keyser's careful, earnest style of writing makes it a pleasure to read his works, and any one who has the "mathematical spirit which is simply the spirit of logical rectitude" will enjoy this unusual essay.

EMMA K. WHITON.

UMI

THE STUDY OF RELIGIONS. By *Stanley A. Cook, M.A.,* Ex-Fellow and Lecturer in the Comparative Study of Religions and in Hebrew and Syriac, Gonville and Caius College, Cambridge. London, A. and C. Black, 1914. Price, 7s. 6d. net.

Mr. Cook is very long-winded, but in spite of dryness and abstractness of style he has written a valuable book. Much thought has evidently gone into it, and its defects are due to a difficult manner of exposition, not to poverty of ideas. This is not an "Introduction" of the type of Jevons's book; it gives no data for the beginner, nor, as one is apt to expect from the title, does it deal chiefly with primitive religion. It is rather the comments of a scholar—Mr. Cook is a recognized authority in his field—on the aims and methods of his study. He has a great deal to say, and much that is extremely good, on the evolution of religion—as is indicated by several chapter headings: Survivals, The Environment and Change, Development and Continuity. "The doctrine of survivals," Mr. Cook says, "is entirely inadequate when it forgets that we are human beings and do not accept beliefs merely because they happen to lie within our reach. The doctrine of survivals, is, in fact, a very handy and cheap explanation of some one else's beliefs and practices—hardly of our own!" Survivals are not simply "left behind," they are subconsciously selected. Mr. Cook warns very wisely against arguing from the part to the whole, against constructing a hypothetical system into which every survival must fit. He warns also against confusing the evolution of beliefs with the evolution of environments, in judging apparent retrogressions. On the critical attitude, on the acceptance of data, Mr. Cook has some excellent observations, and on the historical versus the religious importance of critical revisions. He holds that the present is a time of religious unrest, though like most of us, he cannot point to any definite theology for the future. His conclusion is as follows: "The unbiased student of religions can hardly escape the conviction that the Supreme Power, whom we call God, while enabling man to work out, within limits, his own career, desires the furtherance of those aims and ideals which are for the advance of mankind." η

Just as we are going to press we receive two additional notes from Dr. W. B. Smith to be inserted in his article as indicated respectively on pages 330 and 337.

Page 330: "For which Rutherford's 'nucleus theory,' apparently required by the facts in the scattering of 'alpha rays' (of helium atoms) in passing through laminae, substitutes a positive electric core, extremely minute, for gold only one trillionth of an inch in diameter, in volume one billionth of the atom itself. It would seem that the negative electron is nearly six thousand million times as large as the positive hydrogen core. For Thomson's later views see *Philos. Mag.,* 1913, p. 892."

Page 337: "Why do the members fall together to the center as their energies are dissipated in electric radiation? Bohr (*Philos. Mag.,* 1913, pp. 1, 476, 854) invokes Planck's 'Quantum'-hypothesis in solving this riddle."